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HENRY BRIGGS.

Henry Briggs, born at Halifax in 1556, was educated at St. John's College, Cambridge; he remained in Cambridge, lecturing on mathematics at St. John's, till 1596. In this year, he was appointed the first holder of the Gresham Professorship of Geometry on Gresham's splendid London foundation. In 1619 Sir Henry Savile's chairs of astronomy and geometry were established at Oxford; the Savilian chair of geometry was offered to Briggs, who accepted it and held it till his death in 1630. He thus has the distinction of holding in succession the two earliest of the English chairs of mathematics.

Napier's discovery of logarithms immediately attracted Briggs' attention, and he was soon in correspondence with the famous Scots mathematician, one important topic being the possibility of improving the practical application of logarithms by the introduction of a base and the bringing of logarithms into closer connection with the decimal scale. The final choice of 10 as a base seems due to Napier himself, but he not unnaturally felt unable himself to carry out the computations required, a task which he left to the vigorous and enthusiastic Briggs, who must therefore be regarded as chief after Napier among the popularisers and simplifiers of the logarithmic weapon.

Briggs visited Napier in 1616, two years after the publication of *Mirifici Logarithmorum Canonis Descriptio*, and though the story of their first meeting is well known, it may bear repetition. "On the first publication of Napier's logarithms, Briggs was so surprised with admiration of them that he could have no quietness in himself until he had seen that noble person the Lord Merchiston. He acquaints John Marr herewith, who goes to Scotland before Mr. Briggs purposely to be there when two so learned persons should meet. Mr. Briggs appoints a certain day when to meet in Edinburgh, but failing thereof, the Lord Napier was doubtful he would not come. It happened one day as John Marr and Lord Napier were speaking of Mr. Briggs, 'Ah, John,' saith Merchiston, 'Mr. Briggs will not come.' At the very instant one knocks at the gate. John Marr hastened down and it proved to be Mr. Briggs, whom he brings to my Lord's chamber, where almost one quarter of an hour was spent each beholding the other with admiration, before one spoke."

THE SUM OF THE INTEGRAL PARTS IN AN ARITHMETICAL PROGRESSION.

By J. C. P. Miller

Abstract : The paper gives an exact evaluation of the sum of the integral parts of the terms of an arithmetical progression, with application to the checking of a table of rounded-off multiples of a constant.

1. *Introduction.*

The main purpose of this paper is to derive a formula, suitable for computation, for the sum of the integer parts of the terms of an arithmetical progression, in cases where the common difference has a non-integral part, rational or irrational.

1·1. The principle behind the derivation of the formula may be illustrated by taking a particular example. Consider, for instance, a table of multiples $[n\pi]$, where [ ] denotes integer part. Clearly

$$[n\pi]=3n+(r_1-1) \qquad \text{(1·11)}$$

where r_1 depends on n in such a way that

$$r_1=1 \text{ for } n=0 \text{ to } 7,$$
$$r_1=2 \text{ for } n=8 \text{ to } 14,$$

and so on, the final n for each r_1 being given by

r_1	1	2	3	4 ...	15	16 ...	31	32 ...	47	48 ...		
$a_1=$ last n	7	14	21	28 ...	105	113 ...	218	226 ...	331	339 ...	.	(1·12)

The increase from term to term is usually 7 but occasionally 8. It may be verified that

$$a_1=[r_1x_1]=7r_1+(r_2-1) \qquad \text{(1·13)}$$

where $x_1=1/(\pi-3)$.

Just as (1·12) gives a_1, the last n for which r_1 has the value indicated, so a_2, the last r_1 for which r_2 has a prescribed value, is given in (1·14) below

r_2	1	2	3	4 ...	293	294 ...	587	588 ...	880	881 ...		
$a_2=$ last r_1	15	31	47	63 ...	4687	4702 ...	9390	9405 ...	14077	14092 ...	.	(1·14)

In this case the increase from term to term is usually 16 but occasionally 15, while

$$a_2=[r_2x_2]=16r_2-r_3 \qquad \text{(1·15)}$$

where $x_2=1/(x_1-7)$. The integer nearest to x_2 has been taken as coefficient of r_2 in (1·15); $[x_2]$ could be used equally well, though the resulting steps converge less rapidly.

One further table follows

r_3	1	2	3	4	5	6	7	8	9	10	...
$a_3=$ last r_2	293	587	880	1174	1468	1761	2055	2349	2642	2936	...

with successive increases of either 293 or 294 in a_3. In this case

$$a_3=[r_3x_3]=293r_3+r_4 \qquad \text{(1·16)}$$

with $x_3=1/(16-x_2)$.

The process is based on the continued fraction

$$\pi=3+\frac{1}{7+}\,\frac{1}{16-}\,\frac{1}{293+} \qquad \text{(1·17)}$$

and, incidentally, shows how a constant c may be determined as closely as possible from a rounded-off table of its multiples, in the form

$$c = c_0 \pm \frac{1}{c_1 \pm}\,\frac{1}{c_2 \pm}\,\frac{1}{c_3 \pm}\cdots \qquad \text{.........................(1·18)}$$

1·2. Now, to calculate $S = \sum_{n=1}^{N} [n\pi]$, proceed thus :

From (1·11), $$S = \tfrac{3}{2}N(N+1) + (\Sigma r_1) - N, \qquad \text{.........................(1·21)}$$

Σr_1 is the contribution from the fractional part of π. Consider first a fractional part that is exactly $\frac{1}{7}$, then if $N_1{}^* = [\frac{1}{7}N]$, the corresponding contribution to Σr_1 is

$$7(1+2+3+\dots N_1{}^*) - 1 + (N+1-7N_1{}^*)(N_1{}^*+1)$$
$$= (N_1{}^*+1)(N+1-\tfrac{7}{2}N_1{}^*) - 1. \qquad (1·22)$$

Since the fraction is less than $\frac{1}{7}$, however, it is more convenient to count an integer multiple of $\frac{1}{7}$, that is, its product by an integer $p = 7q$, as $q-1$ rather than as q ; in other words we take $[\frac{1}{7}p]'$, where $[x]'$ denotes the integral part of x, except when x is an exact integer, in which case $[x]' = x - 1$. It is also found necessary to use $N_1 = [(\pi - 3)N]$ rather than $N_1{}^*$, that is to consider the difference of $\Sigma[\frac{1}{7}p]$ from $\Sigma[(\pi-3)N]$, in terms of constants connected with the true sum rather than those connected with the approximation. With these modifications

$$S_1 = \Sigma r_1 = (N_1+1)(N - \tfrac{7}{2}N_1) - S_2. \qquad \text{....................(1·23)}$$

Now, consideration of (1·12) shows that the increases to $r_1 = 16$, to $r_1 = 17$, and so on up to $r_1 = 31$ are each postponed for one value of n, and that all increases to $r_1 = 32$ etc., up to $r_1 = 47$ are postponed two places, and so on. In fact, each increase is postponed by the amount $r_2 - 1$, giving a total loss

$$S_2 = \Sigma(r_2 - 1) = \Sigma r_2 - N_1, \qquad \text{...........................(1·24)}$$

summation being over values of r_1 from 1 to N_1.

Then, from (1·14) and (1·15),

$$(S_2 + N_1) = 16(1+2+\dots+N_2) - 1 + (N_2+1)(N_1+1-16N_2) + \Sigma r_3$$
$$= (N_2+1)(N_1+1-\tfrac{16}{2}N_2) - 1 + \Sigma r_3 \qquad \text{............................(1·25)}$$

where Σr_3, summed over r_2 from 1 to $N_2 \equiv [N_1/c_2]$, can be evaluated by a continuation of this process.

The full statement of the problem is given in §2. Details of the evaluation of $\sum_{n=1}^{N}[nc]$ are given in § 3, and of $\sum_{n=1}^{N}[nc+a]$, with $0 < a < 1$, in § 4, c and a being constants. A numerical illustration is given in § 5.

2. *Statement of the Problem.*

2·1. The sum to be evaluated is

$$S(N\,;\ c, a) = \sum_{r=0}^{N} [rc + a] \qquad \text{..........................(2·11)}$$

in which $[x]$ denotes the integral part of x. It is also necessary to consider the sum

$$S'(N\,;\ c, a) = \sum_{r=0}^{N} [rc + a]' \qquad \text{.........................(2·12)}$$

in which $[x]'$ denotes the integral part of x when x is not an integer, but $[x]' = x - 1$ when x is an integer.

The difference $S - S'$ gives the number of terms which are exact integers; these terms are of considerable interest in the sequel, and were the main source of difficulty in the development of the formulae.

It is of interest to note that if p is any integer, then

$$[x] + [p - x]' = [x]' + [p - x] = p - 1$$

Again, for example, $[2] = 2$ and $[2]' = 1$ correspond to the dual decimal expression of 2 as 2·000 ... and 1·999 ... . The second form is appropriate when we count integer parts by subtraction, thus

$$\sum_1^{10} [\tfrac{1}{2}r] = \sum_1^{10} [r - \tfrac{1}{2}r] = \sum_1^{10} (r - 1 - [\tfrac{1}{2}r]') = 45 - 20 = 25, \text{ since } \sum_1^{10} [\tfrac{1}{2}r]' = 20.$$

In the sequel a_r, b_r, c_r, d_r denote integers.

2·2. It is sufficient to develop the formulae for positive fractional values of c and a, for if

$$c = c_0 + \theta_0 \qquad a = a_0 + \lambda_0 \qquad 0 \leqslant \theta_0, \lambda_0 \leqslant 1 \quad \text{.............(2·21)}$$

then $$[rc + a] = rc_0 + a_0 + [r\theta_0 + \lambda_0] \quad \text{.............(2·22)}$$

and $$S(N\,;\, c, a) = (N+1)(a_0 + \tfrac{1}{2}Nc_0) + S(N\,;\, \theta_0, \lambda_0) \quad \text{.........(2·23)}$$

Similarly

$$S'(N\,;\, c, a) = (N+1)(a_0 + \tfrac{1}{2}Nc_0) + S'(N\,;\, \theta_0, \lambda_0) \quad \text{.........(2·24)}$$

2·3. Again, although not necessary, it is possible, and convenient in computation, to take $0 \leqslant \theta \leqslant \frac{1}{2}$. This follows from the relation

$$[r\theta + \lambda] + [r(1 - \theta) + (1 - \lambda)]' = r \quad \text{.............(2·31)}$$

whence $$S(N\,;\, \theta, \lambda) = \tfrac{1}{2}N(N+1) - S'(N\,;\, 1 - \theta, 1 - \lambda) \quad \text{............(2·32)}$$

and $$S'(N\,;\, \theta, \lambda) = \tfrac{1}{2}N(N+1) - S(N\,;\, 1 - \theta, 1 - \lambda) \quad \text{...............(2·33)}$$

This transformation is not needed when $\theta = 0$, while the case $\lambda = 0$ requires special treatment (see § 3).

2·4. From § 2·2 and § 2·3, with

$$c = d_0 - \phi_0 \quad a = b_0 - \mu_0 \quad 0 < \phi_0, \mu_0 < 1 \quad \text{...............(2·41)}$$

it follows that

$$S(N\,;\, c, a) = (N+1)(b_0 - 1 + \tfrac{1}{2}Nd_0) - S'(N\,;\, \phi_0, \mu_0) \quad \text{............(2·42)}$$

$$S'(N\,;\, c, a) = (N+1)(b_0 - 1 + \tfrac{1}{2}Nd_0) - S(N\,;\, \phi_0, \mu_0) \quad \text{............(2·43)}$$

3. *Development of the formulae. Case* $\lambda = 0$.

3·1. It is convenient to consider first the cases with $\lambda = 0$, which present some special features, due to the fact that $[1 - \lambda] \neq [1 - \lambda]'$ for this particular value of λ. The sums needed are

$$S(N\,;\, \theta) = S(N\,;\, \theta, 0) = \sum_1^N [r\theta], \quad \text{.....................(3·11)}$$

$$S'(N\,;\, \theta) = S'(N\,;\, \theta, 0) = \sum_1^N [r\theta]'. \quad \text{.....................(3·12)}$$

These differ only when one or more terms on the right are exact integers, by the inclusion of an extra unit in (3·11) for each such term.

3·2. The results to be proved are as follows:

If $$N_1 = [N\theta], \quad \frac{1}{\theta} = c_1 + \theta_1 = d_1 - \phi_1, \quad 0 < \theta_1 = 1 - \phi_1 < 1, \quad \text{........(3·21)}$$

then $$S(N\,;\,\theta)=N_1\left(N-c_1\frac{N_1+1}{2}\right)-S'(N_1\,;\,\theta_1)\quad\text{..............}(3{\cdot}22)$$

$$=N_1\left(N+1-d_1\frac{N_1+1}{2}\right)+S(N_1\,;\,\phi_1),\quad\text{.........}(3{\cdot}23)$$

$$S'(N\,;\,\theta)=N_1\left(N-c_1\frac{N_1+1}{2}\right)-S(N_1\,;\,\theta_1),\quad\text{.............}(3{\cdot}24)$$

$$=N_1\left(N+1-d_1\frac{N_1+1}{2}\right)+S'(N_1\,;\,\phi_1),\quad\text{........}(3{\cdot}25)$$

or, if $\theta_1=0$, $$S(N\,;\,1/c_1)=S'(N\,;\,1/c_1)+N_1=N_1\left(N+1-c_1\frac{N+1}{2}\right)\text{....}(3{\cdot}26)$$

Only (3·22) and (3·26) will be proved ; the others follow similarly.

3·3. Denote by μ_p, ν_p respectively the number of terms for which $[r\theta]'<p$, $[r\theta]<p$. Clearly

$$\mu_p\theta\leqslant p<(\mu_p+1)\,\theta,\quad \nu_p\theta<p\leqslant(\nu_p+1)\,\theta,\quad\text{.................}(3{\cdot}31)$$

and $$\mu_p\leqslant\mu_{p+1},\quad \nu_p\leqslant\nu_{p+1}\quad\text{(inequality if } \theta<1)\ \text{............}(3{\cdot}32)$$

while $$\mu_p=\nu_p,\text{ except when } \mu_p\theta=(\nu_p+1)\theta=p,\text{ an integer.}\ \text{.........}(3{\cdot}33)$$

Also, since the number of terms for which $[r\theta]'$ and $[r\theta]$ are at least p are $N-\mu_p$ and $N-\nu_p$ respectively, and since an integer part q contributes a unit to exactly q of these enumerations, it follows that

$$S'(N\,;\,\theta)=\sum_{p=1}^{N_1}(N-\mu_p),\quad S(N\,;\,\theta)=\sum_{p=1}^{N_1}(N-\nu_p),\quad\text{.........}(3{\cdot}34)$$

since a term of (3·11) or (3·12) with integer part q is counted in each of the terms of (3·34) for $p=1(1)q$.

3·4. If $\dfrac{1}{\theta}=c_1$, an integer, so that $\theta_1=0$ in (3·21), then

$$\mu_p=\nu_p+1=c_1p,\quad\text{..}(3{\cdot}41)$$

whence $$S'(N\,;\,1/c_1)=\sum_{p=1}^{N_1}(N-c_1p)=N_1\left(N-c_1\frac{N_1+1}{2}\right),\quad\text{...............}(3{\cdot}42)$$

$$S(N\,;\,1/c_1)=\sum_{p=1}^{N_1}(N+1-c_1p)=N_1\left(N+1-c_1\frac{N_1+1}{2}\right),\quad\text{......}(3{\cdot}43)$$

giving (3·26). Note that $S-S'=N_1=[N/c_1]$, precisely the number of exact integers in the sequence $p\theta$, $p=1(1)N$.

3·5. If $\theta\neq 0$, so that $1/\theta$ lies between the consecutive integers c_1 and $c_1+1=d_1$, then, from the approximations (in defect and in excess)

$$\theta\fallingdotseq\frac{1}{d_1}\quad\text{or}\quad\theta\fallingdotseq\frac{1}{c_1},\quad\text{..............................}(3{\cdot}51)$$

are derived the estimates

$$\mu_p\fallingdotseq pd_1\quad\text{or}\quad\nu_p\fallingdotseq pc_1\ \text{..............................}(3{\cdot}52)$$

The corresponding approximations to (3·34) are

$$\left.\begin{aligned}T_1(N\,;\,\theta)&=\sum_{p=1}^{N_1}(N-pd_1)=N_1\left(N-d_1\frac{N_1+1}{2}\right)\\ T_2(N\,;\,\theta)&=\sum_{p=1}^{N_1}(N-pc_1)=N_1\left(N-c_1\frac{N_1+1}{2}\right)\end{aligned}\right\},\quad\text{.........}(3{\cdot}53)$$

whence (confining attention to (3·22))

$$S(N\,;\ \theta)=\sum_{p=1}^{N_1}(N-\nu_p)=\sum_{p=1}^{N_1}(N-pc_1)+\sum_{p=1}^{N_1}(pc_1-\nu_p)$$

$$=T_2(N\,;\ \theta)+\sum_{p=1}^{N_1}(pc_1-\nu_p). \quad \text{.........................(3·54)}$$

Now, from (3·21) and (3·31)

$$\nu_p-pc_1<p\theta_1\leqslant\nu_p+1-pc_1,$$

so that

$$\nu_p-pc_1=[p\theta_1]', \quad \text{.........................(3·35)}$$

and

$$S(N\,;\ \theta)=N_1\left(N-c_1\frac{N_1+1}{2}\right)-\sum_{p=1}^{N_1}[p\theta_1]'$$

$$=N_1\left(N-c_1\frac{N_1+1}{2}\right)-S'(N_1\,;\ \theta_1), \quad \text{....(3·56)}$$

which is (3·22).

The remaining formulae (3·23) to (3·25) may be derived similarly.

3·6. By continued application of (3·22) to (3·26), the sums $S(N,\ \theta)$, $S'(N,\ \theta)$ are readily evaluated. For example, if

$$\theta=\frac{1}{c_1+}\,\frac{1}{d_2-}\,\frac{1}{c_3+}\,\frac{1}{d_4-}\,\frac{1}{d_5-}\cdots, \quad \text{.........................(3·61)}$$

and if

$$\frac{1}{\theta}=c_1+\theta_1,\quad \frac{1}{\theta_1}=d_2-\phi_2,\quad \frac{1}{\phi_2}=c_3+\theta_3,\quad \frac{1}{\theta_3}=d_4-\phi_4,\quad \frac{1}{\phi_4}=d_5-\phi_5,\ \ldots\text{(3·62)}$$

while

$$N_1=[N\theta],\quad N_2=[N_1\theta_1],\quad N_3=[N_2\phi_2],\quad N_4=[N_3\theta_3],\quad N_5=[N_4\phi_4],\ \ldots\text{(3·63)}$$

then

$$S(N\,;\ \theta)=N_1\left(N-c_1\frac{N+1}{2}\right)-N_2\left(N_1+1-d_2\frac{N_2+1}{2}\right)-N_3\left(N_2-c_3\frac{N_3+1}{2}\right)$$

$$+N_4\left(N_3+1-d_4\frac{N_4+1}{2}\right)+N_5\left(N_4+1-d_5\frac{N_5+1}{2}\right)+\ldots\ \text{.......(3·64)}$$

3·7. It should be noted that there is a (1, 1)-correspondence between exact integer terms in the two sequences $r\theta$, $r=1\,(1)N$ and $p\theta_1$ or $p\phi_1$, $p=1\,(1)N_1$. For if, say, $r\theta=p$, then $r=\mu_p=\nu_p+1$ by (3·33).

Hence

$$\mu_p/(c_1+\theta_1)=\mu_p/(d_1-\phi_1)=p \quad \text{.........................(3·71)}$$

and

$$p\theta_1=\mu_p-pc_1,\quad p\phi_1=pd_1-\mu_p \quad \text{.........................(3·72)}$$

both integers. The process is clearly reversible.

Hence

$$S(N\,;\theta)-S'(N\,;\theta)=S(N_1\,;\theta_1)-S'(N_1\,;\theta_1)=S(N_1\,;\phi_1)-S'(N_1\,;\phi_1)\ \ \ldots\text{(3·73)}$$

and the exact integers, if they occur, are evidently transmitted, through successive applications of (3·22) to (3·26), until a final term $S(N_s\,;\theta_s)$ or $S'(N_s\,;\theta_s)$ is reached with $\theta_s=1/c_{s+1}$, i.e. such that $\theta_{s+1}=0$.

4. *Development of the formulae. General Case* $\lambda\neq 0$.

4·1. The sums to be evaluated are

$$S(N\,;\ \theta,\lambda)=\sum_{1}^{N}[r\theta+\lambda], \quad \text{.........................(4·11)}$$

$$S'(N\,;\ \theta,\lambda)=\sum_1^N[r\theta+\lambda]', \qquad\text{(4·12)}$$

and the results to be proved are :

If $N_1=[N\theta+\lambda]$, $\dfrac{1}{\theta}=c_1+\theta_1=d_1-\phi_1$,

$$\frac{1-\lambda}{\theta}=a_1+\lambda_1=b_1-\kappa_1,\quad a_1,\,b_1,\,c_1,\,d_1 \text{ integers}, \qquad\text{(4·13)}$$

where $0\leqslant\theta_1<1$, $0<\phi_1<1$, $0\leqslant\lambda_1<1$, $0<\kappa_1<1$, so that $d_1=c_1+1$, $b_1=a_1+1$,

then
$$S(N\,;\ \theta,\lambda)=N_1\left(N-a_1-c_1\frac{N_1-1}{2}\right)-S'(N_1-1\,;\ \theta_1,\lambda_1), \qquad\text{(4·14)}$$

$$=N_1\left(N-a_1-d_1\frac{N_1-1}{2}\right)+S(N_1-1\,;\ \phi_1,\kappa_1), \qquad\text{(4·15)}$$

$$S'(N\,;\ \theta,\lambda)=N_1\left(N-a_1-c_1\frac{N_1-1}{2}\right)-S(N_1-1\,;\ \theta_1,\lambda_1), \qquad\text{(4·16)}$$

$$=N_1\left(N-a_1-d_1\frac{N_1-1}{2}\right)+S'(N_1-1\,;\ \phi_1,\kappa_1), \qquad\text{(4·17)}$$

or, if $\theta_1=0$, $\theta=1/c_1$

$$S\left(N\,;\ \frac{1}{c_1},\lambda\right)=S'\left(N\,;\ \frac{1}{c_1},\lambda\right)+N_1=N_1\left(N+1-a_1-c_1\frac{N_1-1}{2}\right), \qquad\text{(4·18)}$$

if exact integer multiples occur in the sequence, otherwise

$$S\left(N\,;\ \frac{1}{c_1},\lambda\right)=S'\left(N\,;\ \frac{1}{c_1},\lambda\right)=N_1\left(N-a_1-c_1\frac{N_1-1}{2}\right). \qquad\text{(4·19)}$$

4·2. Denote, as in § 3·3, by μ_p and ν_p the number of terms for which $[r\theta+\lambda]'<p$, $[r\theta+\lambda]<p$ respectively. These are given by

$$\mu_p\theta+\lambda\leqslant p<(\mu_p+1)\theta+\lambda,\quad \nu_p\theta+\lambda<p\leqslant(\nu_p+1)\theta+\lambda, \qquad\text{(4·21)}$$

while, as before

$$\mu_p\leqslant\mu_{p+1},\quad \nu_p\leqslant\nu_{p+1}\quad (\text{inequality if } \theta<1), \qquad\text{(4·22)}$$

and $\mu_p=\nu_p$, except when $\mu_p\theta+\lambda=(\nu_p+1)\theta+\lambda=p$, an integer .... (4·23)

Then, as before

$$S'(N\,;\ \theta,\lambda)=\sum_{p=1}^{N_1}(N-\mu_p),\quad S(N\,;\ \theta,\lambda)=\sum_{p=1}^{N_1}(N-\nu_p). \qquad\text{(4·24)}$$

4·3. Consider first the case $1/\theta=c_1$ an integer. In this case, with $\lambda\neq 0$, exact integer terms do not necessarily occur.

If
$$\frac{a_1}{c_1}+\lambda<1<\frac{a_1+1}{c_1}+\lambda, \qquad\text{(4·31)}$$

so that
$$\mu_p=\nu_p=a_1+(p-1)c_1 \qquad\text{(4·32)}$$

then, by (4·24),
$$S\left(N\,;\ \frac{1}{c_1},\lambda\right)=S'\left(N\,;\ \frac{1}{c_1},\lambda\right)=\sum_{p=1}^{N_1}(N-a_1-\overline{p-1}\,c_1)$$

$$=N_1\left(N-a_1-c_1\frac{N_1-1}{2}\right), \qquad\text{(4·33)}$$

which is (4·19).

If
$$\frac{a_1}{c_1}+\lambda=1, \quad\text{.............................}(4\cdot34)$$
so that
$$\mu_p=\nu_p+1=a_1+(p-1)c_1 \quad\text{.........................}(4\cdot35)$$
then
$$S\left(N\,;\,\frac{1}{c_1},\lambda\right)=S'\left(N\,;\,\frac{1}{c_1},\lambda\right)+N_1=\sum_{p=1}^{N_1}(N_1+1-a_1-\overline{p-1}\,c_1)$$
$$=N_1\left(N+1-a_1-c_1\frac{N_1-1}{2}\right), \quad\text{..................}(4\cdot36)$$
which is (4·18).

4·4. If, on the other hand
$$\frac{1}{\theta}=c_1+\theta_1=d_1-\psi_1, \quad 0<\theta_1=1-\phi_1<1, \quad\text{................}(4\cdot41)$$
then, as in (3·52)
$$\mu_p \doteqdot a_1+(p-1)d_1 \quad\text{or}\quad \nu_p \doteqdot a_1+(p-1)c_1, \quad\text{...............}(4\cdot42)$$
in which
$$a_1\theta+\lambda\leqslant 1<(a_1+1)\theta+\lambda \quad\text{..........................}(4\cdot43)$$
give estimates to μ_p and ν_p.

The corresponding approximations to (4·24) are
$$T_1(N\,;\,\theta,\lambda)=\sum_{p=1}^{N_1}(N-a_1-d_1\overline{p-1})=N_1\left(N-a_1-d_1\frac{N_1-1}{2}\right), \quad(4\cdot44)$$
$$T_2(N\,;\,\theta,\lambda)=\sum_{p=1}^{N_1}(N-a_1-c_1\overline{p-1})=N_1\left(N-a_1-c_1\frac{N_1-1}{2}\right), \quad(4\cdot45)$$
whence, taking a particular case as an illustration,
$$S'(N\,;\,\theta,\lambda)=\sum_{p=1}^{N_1}(N-\mu_p)=\sum_{p=1}^{N_1}(N-a_1-d_1\overline{p-1})+\sum_{p=1}^{N_1}(a_1+d_1\overline{p-1}-\mu_p). \quad(4\cdot46)$$

Now, from (4·13) and (4·21)
$$\mu_p\leqslant(p-\lambda)(d_1-\phi_1)<\mu_p+1,$$
while
$$a_1+(p-1)d_1=(1-\lambda)(d_1-\phi_1)-\lambda_1+(p-1)d_1. \quad\text{.........}(4\cdot47)$$
whence
$$a_1+(p-1)d_1+1=(p-\lambda)(d_1-\phi_1)+(p-1)\phi_1+1-\lambda_1=a_1+1+(p-1)d_1$$
Hence $a_1+(p-1)d_1-\mu_p<(p-1)\phi_1+(1-\lambda_1)\leqslant a_1+1+(p-1)d_1-\mu_p$

so that
$$a_1+(p-1)d_1-\mu_p=[(p-1)\phi_1+(1-\lambda_1)]'. \quad\text{...............}(4\cdot48)$$
This, using (4·46) gives (4·17). Formulae (4·14) to (4·16) may be derived similarly.

As in §3, repeated application of (4·14) to (4·18) with a possible final step by means of (3·22) to (3·26), yields the value of any $S(N\,;\,c,a)$.

5. *A numerical example.*

As an illustration of the application of the formulae consider $S(100\,;\,\frac{106}{39},\frac{1}{3})$ and $S'(100\,;\,\frac{106}{39},\frac{1}{3})$. The values by direct summation were found to be 13710 and 13708. The difference, 2, is the number of integral values of $\frac{106}{39}p+\frac{1}{3}$ for $p\leqslant 100$. In fact
$$26\times\tfrac{106}{39}+\tfrac{1}{3}=71, \quad 65\times\tfrac{106}{39}+\tfrac{1}{3}=177.$$

Rational values of c and a have been chosen so that use of the formulae of § 3 may be demonstrated. These are needed only for the final stage; for the earlier

stages and for irrational c and a, or for cases when the denominator of c is *not* a multiple of the denominator of a, only the formulae of §4 are needed.

Two derivations of the sum are given, corresponding to the two continued fractions

$$c=2+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5}=3-\frac{1}{4-}\,\frac{1}{2+}\,\frac{1}{5}.$$

A third development $c=2+\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{3-}\,\frac{1}{6}$ may be used as an exercise.

$$N=100 \qquad c=c_0+\theta_0=2+\tfrac{28}{39} \qquad a=a_0+\lambda_0=0+\tfrac{1}{3}$$

$$S(100\,;\,\tfrac{106}{39},\tfrac{1}{3})-S(100\,;\,\tfrac{28}{39},\tfrac{1}{3})=10100$$

$$N_1=[N\theta_0+\lambda_0]=72 \qquad \frac{1}{\theta_0}=c_1+\theta_1=1+\tfrac{11}{28} \qquad \frac{1-\lambda_0}{\theta_0}=a_1+\lambda_1=0+\tfrac{13}{14}$$

$$S(100\,;\,\tfrac{28}{39},\tfrac{1}{3})+S'(71\,;\,\tfrac{11}{28},\tfrac{13}{14})=4644$$

$$N_2=[(N_1-1)\theta_1+\lambda_1]=28 \qquad \frac{1}{\theta_1}=c_2+\theta_2=2+\tfrac{6}{11} \qquad \frac{1-\lambda_1}{\theta_1}=a_2+\lambda_2=0+\tfrac{2}{11}$$

$$S'(71\,;\,\tfrac{11}{28},\tfrac{13}{14})+S(27\,;\,\tfrac{6}{11},\tfrac{2}{11})=1232$$

$$N_3=[(N_2-1)\theta_2+\lambda_2]=14 \qquad \frac{1}{\theta_2}=c_3+\theta_3=1+\tfrac{5}{6} \qquad \frac{1-\lambda_2}{\theta_2}=a_3+\lambda_3=1+\tfrac{1}{2}$$

$$S(27\,;\,\tfrac{6}{11},\tfrac{2}{11})+S'(13\,;\,\tfrac{5}{6},\tfrac{1}{2})=273$$

$$N_4=[(N_3-1)\theta_3+\lambda_3]=11 \qquad \frac{1}{\theta_3}=c_4+\theta_4=1+\tfrac{1}{5} \qquad \frac{1-\lambda_3}{\theta_3}=a_4+\lambda_4=0+\tfrac{3}{5}$$

$$S'(13\,;\,\tfrac{5}{6},\tfrac{1}{2})+S(10\,;\,\tfrac{1}{5},\tfrac{3}{5})=88$$

$$N_5=[(N_4-1)\theta_4+\lambda_4]=2 \qquad \frac{1}{\theta_4}=c_5\quad=5 \qquad \frac{1-\lambda_4}{\theta_4}=a_5\quad=2$$

$$S(10\,;\,\tfrac{1}{5},\tfrac{3}{5})=13$$

whence $\qquad S(100\,;\,\tfrac{106}{39},\tfrac{1}{3})=13710$

$$N=100 \qquad c=d_0-\phi_0=3-\tfrac{11}{39} \qquad a=b_0-\kappa_0=1-\tfrac{2}{3}$$

$$S(100\,;\,\tfrac{106}{39},\tfrac{1}{3})+S'(100\,;\,\tfrac{11}{39},\tfrac{2}{3})=15150$$

$$N_1=[N\phi_0+\kappa_0]=28 \qquad \frac{1}{\phi_0}=d_1-\phi_1=4-\tfrac{5}{11} \qquad \frac{1-\kappa_0}{\phi_0}=b_1-\kappa_1=2-\tfrac{9}{11}$$

$$S'(100\,;\,\tfrac{11}{39},\tfrac{2}{3})-S'(27\,;\,\tfrac{5}{11},\tfrac{9}{11})=1260$$

$$N_2=[(N_1-1)\phi+\kappa_1]=13 \qquad \frac{1}{\phi_1}=c_2+\theta_2=2+\tfrac{1}{5} \qquad \frac{1-\kappa_1}{\phi_1}=a_2+\lambda_2=0+\tfrac{2}{5}$$

$$S'(27\,;\,\tfrac{5}{11},\tfrac{9}{11})+S(12\,;\,\tfrac{1}{5},\tfrac{2}{5})=195$$

$$N_3=[(N_2-1)\theta_2+\lambda_2)=2 \qquad \frac{1}{\theta_2}=c_3\quad=5 \qquad \frac{1-\lambda_1}{\theta_2}=a_3\quad=3$$

$$S(12\,;\,\tfrac{1}{5},\tfrac{2}{5})=15$$

whence, as before $\qquad S(100\,;\,\tfrac{106}{39},\tfrac{1}{3})=13710$

6. *Application to sums of rounded-off multiples of a constant.*

6·1. When applying the formulae of the preceding paragraphs to the determination of the sum of equally-spaced multiples of a constant, rounded-off to

a definite number of decimals, take the final digit as unit (that is, if n-decimal values are given multiply the constant by 10^n and round-off to the nearest integer).

6·2. If the constant is c, it is clear that $\sum_{n=1}^{N} [nc]$ represents the sum obtained by rounding *down*, while $\sum_{n=1}^{N} [nc+1]'$ represents the sum obtained by rounding *up*.

The result of rounding-off to the nearest integer may be obtained in two ways

(*a*) $\sum_{n=1}^{N} [nc+\frac{1}{2}]=S(N\,;\ c,\ \frac{1}{2})$ gives the sum if exact half-integers are rounded *up*, while $S'(N\,;\,c,\,\frac{1}{2})$ gives the result if these are rounded *down*.

(*b*) Alternatively, it may be noted that, where $x=a+\lambda$, a an integer, $0 \leqslant \lambda < 1$, and if

$$\tfrac{1}{2} \leqslant \lambda < 1, \text{ then } [2x]-[x]=2a+1-a=a+1,$$

while if

$$0 \leqslant \lambda < \tfrac{1}{2}, \text{ then } [2x]-[x]=2a-a=a,$$

Thus $[2x]-[x]$ gives the rounded-off value of x, hence

$$\sum_{n=1}^{N} \{[2nc]-[nc]\}=S(N\,;\ c,\ \tfrac{1}{2})$$

Similarly

$$\sum_{n=1}^{N} \{[2nc]'-[nc]'\}=S'(N\,;\ c,\ \tfrac{1}{2})$$

The two approaches involve similar amounts of calculation, in (*a*) the c_r, λ_r have both to be calculated, while in (*b*) the c_r for both c and $2c$ are needed.

6·3. The previous paragraph covers cases where exact halves are rounded systematically all up or all down. A common rule is to round off such cases to the nearest *even* number, this has two advantages (*a*) the final 5 is not used, thus avoiding ambiguities in some cases if a further rounding-off is desired (*b*) the number of items rounded-up is, on the average, equal to the number rounded down.

Two expressions to cover this type of rounding-off are the following

$$\sum_{n=1}^{N} \{[nc+\tfrac{1}{2}]'+[\tfrac{1}{2}nc+\tfrac{1}{4}]-[\tfrac{1}{2}nc+\tfrac{1}{4}]'\}=\sum_{n=1}^{N} \{[\tfrac{1}{2}nc+\tfrac{1}{4}]+[\tfrac{1}{2}nc+\tfrac{3}{4}]'\}.$$

The former is simply equivalent to rounding-down, with special counting of cases where $nc+\frac{1}{2}$ is an even integer, and the addition of an appropriate number of units. This is probably the simplest approach, since exact integer values of $(nc+\frac{1}{2})$ must be alternately odd and even, and are fairly easy to count.

6·4. The methods outlined in this section are convenient only when N is large—for short runs a direct check of individual multiples is simpler. On the other hand, if N is large, the chance of compensating errors is higher. To overcome this, when N is very large, say $N=PQ$, P and Q both being reasonably large integers, the two sets of sums (in which the term for $N=PQ$ is omitted for convenience).

$$\sum_{p=0}^{P-1} [(pQ+q)c], \qquad q=0(1)Q-1,$$

and

$$\sum_{p'=0}^{Q-1} [(p'P+q')c], \quad q'=0(1)P-1,$$

may be evaluated and checked. Errors, if they are present, may then be readily tracked down, and the chance of compensating errors is negligible.

As an illustration, consider the table of conversion from degrees and minutes of arc to radians given in Chambers's Seven-figure Mathematical Tables pp. 251–262. These may be added down (sums of 60 terms) and across (sums of 90 terms). This has been tried out on the first four columns and first two rows, yielding two discrepancies which have been traced to errors in the tables at 1° 10′ and at 31° 1′. The work of computing the sums in such cases is simplified if the method of 6·2 (a) is used, since N and c are the same for all columns (or for all rows), only a changing from column to column. In other words, if κ is the conversion factor, d the number of degrees and m the number of minutes, the sums by columns are $\sum_m \kappa(d+m)$ with only the constant κd changing from column to column, while the sums by rows are $\sum_d \kappa(m+d)$ with the constant κm changing from row to row.

My thanks are due to Mr. H. E. Salzer, of the Computation Laboratory, National Bureau of Standards, Washington, D.C., who read through this paper with great care and pointed out a number of inconsistencies in the original version. J. C. P. M.

GLEANINGS FAR AND NEAR.

1709. For all its ramifications and applications and philosophic implications, the mathematics of probability still depends on the dice-player's faith that seven will turn up three times as often as two, because one plus six, two plus five, and three plus four make seven, while only one plus one make two.—Gerard Piel, "Mathematics Comes Out of the Classroom," *Yale Review*, Autumn 1949, pp. 132–141. [Per Prof. H. D. Larsen.]

1710. Mr. Roger Lubbock is no doubt right when he indicates that a motorist travelling at 50 m.p.h. may avoid collision with a swerving cyclist more easily than one travelling at 20 or 25 m.p.h. But has he not overlooked a very important element in the case—namely, that if there is a collision, it will be some six to four times more severe in the first case than in the others —since impact is proportioned to the square of the velocity—quite probably making the difference between killing the cyclist and only hurting him.—*The Times*, November 4, 1950. [Per Professor H. R. Hassé.]

1711. Problème. Bacchus trouve Sylène endormi près d'un tonneau plein de vin, et boit pendant les trois cinquièmes du temps que Sylène aurait employé à vider le tonneau. Sylène s'éveille et boit le reste du vin. Si Bacchus et Sylène eussent bu ensemble, le tonneau eût été vidé six heures plus tôt, et Bacchus n'aurait bu que les deux tiers de ce qu'il a laissé à Sylène. On demande combien il faudrait d'heures à chacun d'eux en particulier, pour vider le tonneau.—A. A. L. Reynaud et J. M. C. Duhamel, *Problèmes et dèveloppemens sur diverses parties des mathématiques* (Paris, 1823), p. 177.

The authors take x and $5y$ for the numbers of hours that Bacchus and Silenus, drinking separately, would each take to empty the cask, and they obtain the equations

$$25xy^2 - 75y^3 + 3x^2y - 6x^2 - 30xy = 0,$$

$$30y^2 + 11xy - 2x^2 = 0\,;$$

the relevant system of solutions is $x = 15$, $y = 2$. It is much to be regretted that this problem is the only one of its kind in the book. [Per Prof. G. N. Watson.]

CYCLIC PENTAGONS.

By F. Bowman.

1. Typical cyclic configurations of five-bar linkages are shown in Figs. 1, 2, 3, 4. It is proposed to solve the problem of finding all the cyclic configurations of a five-bar linkage of which the lengths of the bars are given. It will be shown that the solution depends upon an equation of the seventh degree which can be expressed in a very simple form.

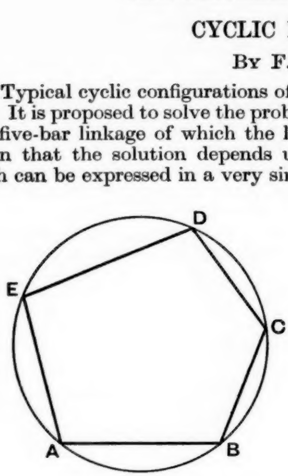

Fig. 1.

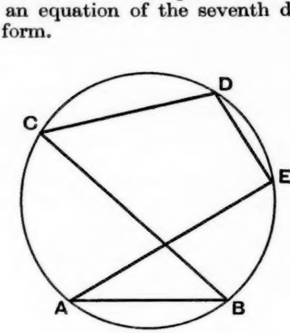

Fig. 2.

2. Let the lengths of the bars be given by $a_1 = AB$, $a_2 = BC$, $a_3 = CD$, $a_4 = DE$, $a_5 = EA$, and let the bars subtend angles $2\theta_1$, $2\theta_2$, $2\theta_3$, $2\theta_4$, $2\theta_5$ respectively at the centre of the circumcircle. Except in Fig. 1, the order of the vertices on the perimeter of the linkage is not the same as the order on the circumference of the circle. Let the angles $2\theta_1$, $2\theta_2$, ... be measured from A to B, B to C, ... all in the same sense round the circle. Let each of these angles lie between $0°$ and $360°$, and let $\Sigma 2\theta$ denote their sum. Then, if the angles are measured in the usual positive sense, $\Sigma 2\theta = 360°$ in Fig. 1, and $\Sigma 2\theta = 3 \times 360°$ in Figs. 2, 3, 4; but if the angles are measured in the opposite sense, then $\Sigma 2\theta = 4 \times 360°$

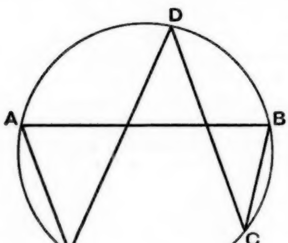

Fig. 3.

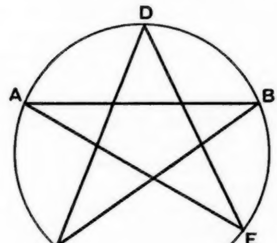

Fig. 4.

in Fig. 1, and $\Sigma 2\theta = 2 \times 360°$ in Figs. 2, 3, 4. Thus the sum of the angles is an odd or an even multiple of $360°$, according as they are measured in one sense or the other. We shall suppose in what follows that the sense is chosen so that the sum of the angles is an odd multiple of $360°$, and hence so that

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 = 180° \text{ or } 3 \times 180°, \quad \text{.....................(1)}$$

this choice being made because it agrees with the usual way of measuring the angles in the ordinary convex case.

3. When the five bars are given, twelve linkages can be constructed by permuting the bars in all possible ways. Let the interchange of two adjacent bars be called an " alternation ". Then any one permutation can be derived from any other by a succession of alternations. Now when an alternation is applied to a cyclic configuration, one of the two possible new configurations will evidently be cyclic with the circumcircle unchanged. Consequently, any possible circumcircle of any one linkage is a circumcircle for all the twelve linkages. Moreover, all twelve linkages, when inscribed in the same circle, have the same area, for this can be regarded as the sum of the areas of the five triangles which the bars form with the radii that pass through the joints of the linkage.

4. Consider the diagonals of one of the set of twelve configurations which can be inscribed in the same circle, the simplest set being the one in which every linkage is convex. In every configuration there are five diagonals, each of which is one side of a triangle of which the other two sides are two bars of the linkage. Since two bars can be chosen out of five in ten ways, it follows that, of the $5 \times 12 = 60$ diagonals that occur in the twelve configurations, only ten are distinct. These ten distinct diagonals are shown in Figs. 5 and 6, in which two of the twelve convex configurations are drawn. In Fig. 5 the order of the links is a_1, a_2, a_3, a_4, a_5; in Fig. 6 it is a_1, a_3, a_5, a_2, a_4. The notation

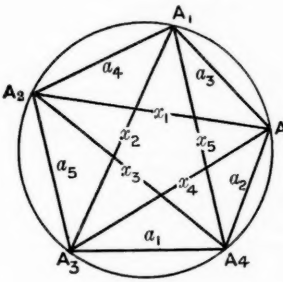

FIG. 5.

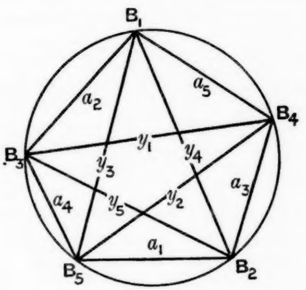

FIG. 6.

adopted is such that the diagonal which has neither of its ends in common with the link a_r is denoted by x_r in Fig. 5, by y_r in Fig. 6. The other ten convex configurations are not drawn; five of them can be derived from Fig. 5, each by a single alternation, the result of which is a figure in which the diagonals consist of three x's and two y's; the other five are derivable from Fig. 6, each by a single alternation, the result of which is a figure in which the diagonals consist of three y's and two x's.

5. We now define the diagonals and other magnitudes analytically in terms of the angles $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, so that the definitions may apply to any cyclic configuration, whether convex or not.

Let r, assumed positive, denote the radius of the circumcircle. The sides, all positive, expressed in terms of r and the angles, are given by

$$a_1 = 2\,r \sin \theta_1,\ a_2 = 2\,r \sin \theta_2,\ \dots\ ,\ a_5 = 2\,r \sin \theta_5 \quad \dots\dots\dots\dots\dots(2)$$

We define the ten diagonals, which will be positive or negative, as the case may be, by

$$x_1 = 2\,r \sin(\theta_3 + \theta_4),\ x_2 = 2\,r \sin(\theta_4 + \theta_5), \dots \qquad (3)$$

$$y_1 = 2\,r \sin(\theta_2 + \theta_5),\ y_2 = 2\,r \sin(\theta_1 + \theta_3), \dots \qquad (4)$$

We define fifteen cosines, $l_1, l_2, \dots l_5$; $m_1, m_2, \dots, m_5$; $n_1, n_2, \dots n_5$ by

$$l_1 = \cos\theta_1, \qquad l_2 = \cos\theta_2, \dots\dots \qquad (5)$$

$$m_1 = \cos(\theta_3 + \theta_4),\ m_2 = \cos(\theta_4 + \theta_5), \dots\dots \qquad (6)$$

$$n_1 = \cos(\theta_2 + \theta_5),\ n_2 = \cos(\theta_1 + \theta_3), \dots\dots \qquad (7)$$

so that $rl_1, rl_2, \dots$; $rm_1, rm_2, \dots$; $rn_1, rn_2, \dots$ will be the (signed) perpendiculars from the centre of the circle on the sides and diagonals respectively.

We define the area S, " enclosed " by the linkage, by

$$S = \tfrac{1}{2} r^2 \sin 2\theta_1 + \tfrac{1}{2} r^2 \sin 2\theta_2 + \dots + \tfrac{1}{2} r^2 \sin 2\theta_5, \qquad \dots\dots\dots\dots(8)$$

which can also be written

$$S = \tfrac{1}{2} r(a_1 l_1 + a_2 l_2 + \dots + a_5 l_5) \qquad \dots\dots\dots\dots(9)$$

6. Referring to Figs. 5 and 6 we note that

$$l_1 = \cos\theta_1 = \cos A_4A_1A_3 = \cos B_2B_1B_5$$

and hence that l_1 is given in terms of diagonals by either of the formulae

$$2x_2x_5l_1 = x_2^2 + x_5^2 - a_1^2, \quad 2y_3y_4l_1 = y_3^2 + y_4^2 - a_1^2 ; \qquad \dots\dots\dots\dots(10)$$

similar formulae will give l_2, l_3, l_4, l_5.

Again, we have $\qquad m_1 = \cos(\theta_3 + \theta_4) = -\cos A_1$

and hence m_1 is given by

$$2a_3a_4m_1 = x_1^2 - a_3^2 - a_4^2. \qquad \dots\dots\dots\dots(11)$$

Similarly, n_1 is given by

$$2a_2a_5n_1 = y_1^2 - a_2^2 - a_2^2, \qquad \dots\dots\dots\dots(12)$$

and similar formulae will give m_2, m_3, m_4, m_5 ; n_2, n_3, n_4, n_5.

7. Next, in Fig. 5 consider the convex quadrilateral of which the sides are a_5, a_1, a_2, x_1. By Ptolemy's theorem, the product of the diagonals is equal to the sum of the products of pairs of opposite sides : that is, $x_3x_4 = a_1x_1 + a_2a_5$. When the sides a_1 and a_2 are interchanged, we obtain a quadrilateral in which the diagonals are x_4 and y_1, and hence $x_4y_1 = a_2x_1 + a_1a_5$; by interchanging a_1 and a_5 we should have obtained a quadrilateral with diagonals x_3 and y_1, and hence $x_3y_1 = a_5x_1 + a_1a_2$. We thus obtain three Ptolemy equations from the quadrilateral of sides a_5, a_1, a_2, x_1 ; and in the same kind of way we could obtain three more from each of the other four quadrilaterals bounded by three adjacent sides and a diagonal. In all, we thus obtain $5 \times 3 = 15$ Ptolemy equations from Fig. 5, and similarly we should find 15 more from Fig. 6. These $2 \times 15 = 30$ equations are set out below, and by the same equations the 30 symbols $\lambda_1, \dots, \Lambda_1, \dots$ are defined.

$y_1x_3 = a_5x_1 + a_1a_2 = \lambda_1$	(1·1)	$x_1y_5 = a_4y_1 + a_1a_3 = \Lambda_1$	(6·1)
$x_3x_4 = a_1x_1 + a_2a_5 = \mu_1$	(1·2)	$y_5y_2 = a_1y_1 + a_3a_4 = M_1$	(6·2)
$x_4y_1 = a_2x_1 + a_5a_1 = \nu_1$	(1·3)	$y_2x_1 = a_3y_1 + a_4a_1 = N_1$	(6·3)
$y_2x_4 = a_1x_2 + a_2a_3 = \lambda_2$	(2·1)	$x_2y_1 = a_5y_2 + a_2a_4 = \Lambda_2$	(7·1)
$x_4x_5 = a_2x_2 + a_3a_1 = \mu_2$	(2·2)	$y_1y_3 = a_2y_2 + a_4a_5 = M_2$	(7·2)
$x_5y_2 = a_3x_2 + a_1a_2 = \nu_2$	(2·3)	$y_3x_2 = a_4y_2 + a_5a_2 = N_2$	(7·3)

$$
\begin{array}{llcl|llcl}
y_3x_5 = a_2x_3 + a_3a_4 = \lambda_3 & & (3{\cdot}1) & & x_3y_2 = a_1y_3 + a_3a_5 = \Lambda_3 & & (8{\cdot}1) \\
x_5x_1 = a_3x_3 + a_4a_2 = \mu_3 & & (3{\cdot}2) & & y_2y_4 = a_3y_3 + a_5a_1 = M_3 & & (8{\cdot}2) \\
x_1y_3 = a_4x_3 + a_2a_3 = \nu_3 & & (3{\cdot}3) & & y_4x_3 = a_5y_3 + a_1a_3 = N_3 & & (8{\cdot}3) \\
y_4x_1 = a_3x_4 + a_4a_5 = \lambda_4 & & (4{\cdot}1) & & x_4y_3 = a_2y_4 + a_4a_1 = \Lambda_4 & & (9{\cdot}1) \\
x_1x_2 = a_4x_4 + a_5a_3 = \mu_4 & & (4{\cdot}2) & & y_3y_5 = a_4y_4 + a_1a_2 = M_4 & & (9{\cdot}2) \\
x_2y_4 = a_5x_4 + a_3a_4 = \nu_4 & & (4{\cdot}3) & & y_5x_4 = a_1y_4 + a_2a_4 = N_4 & & (9{\cdot}3) \\
y_5x_2 = a_4x_5 + a_5a_1 = \lambda_5 & & (5{\cdot}1) & & x_5y_4 = a_3y_5 + a_5a_2 = \Lambda_5 & & (10{\cdot}1) \\
x_2x_3 = a_5x_5 + a_1a_4 = \mu_5 & & (5{\cdot}2) & & y_4y_1 = a_5y_5 + a_2a_3 = M_5 & & (10{\cdot}2) \\
x_3y_5 = a_1x_5 + a_4a_5 = \nu_5 & & (5{\cdot}3) & & y_1x_5 = a_2y_5 + a_3a_5 = N_5 & & (10{\cdot}3)
\end{array}
$$

8. When the sides $a_1, a_2, \ldots$ and the diagonals $x_1, x_2, \ldots, y_1, y_2, \ldots$ in the last thirty equations are expressed in terms of the radius r and the angles $\theta_1, \theta_2, \ldots$ they become identities in the angles, in virtue of (1). Consequently, they apply not only to the twelve convex cyclic configurations, but also to any set of twelve cyclic configurations with a common circumcircle. This applies also to the equations of § 6 when, in addition to the sides and diagonals, the cosines $l_1, m_1, n_1, \ldots$ are expressed in terms of the angles.

If the angles $\theta_1, \theta_2, \ldots$ were measured in the sense in which their sum is an even multiple of $180°$, the result would be merely to change the sign of each x, each y and each l, because each θ would be replaced by its supplement.

9. The set of thirty equations of § 7 can easily be solved for the ten unknown diagonals in terms of the five sides. There is no difficulty in selecting ten equations upon which the remaining twenty can be shown to be linearly dependent, thus verifying that the thirty equations are compatible.

It will be shown in § 10 that the diagonal x_1 satisfies an equation of the seventh degree, the coefficients of which are functions of the sides, and that every other diagonal can be expressed as a rational function of x_1 and the sides, provided $x_1 \neq 0$. It will then follow that the cosines $l_1, m_1, n_1, \ldots$ are expressible in the same kind of way, and that the same is true of r^2 and rS. The diagonal x_1 could, of course, be replaced in these statements by any other diagonal.

When the seven values of r^2 are all real and positive, there will be seven circles in which the five-bar linkage can be inscribed. In each circle can be inscribed all the twelve linkages which can be derived by permuting the links. Thus, at most, there will be $7 \times 12 = 84$ cyclic configurations of the five-bar linkages constructed from five given bars.

The seven circles may not all be real, but there will always be one real circle, the one in which the twelve linkages are all convex. The convex cyclic configuration can be regarded as the configuration of the freely-jointed linkage when it is in equilibrium under the action of a uniform internal pressure.*

10. To find the equation satisfied by x_1 we have, from the first and sixth of the sets of three equations in § 7,

$$y_1^2 = \lambda_1\nu_1/\mu_1, \quad x_1^2 = \Lambda_1 N_1/M_1, \qquad \text{.......................(13)}$$

or,

$$(a_1x_1 + a_2a_5)\,y_1^2 = (a_5x_1 + a_1a_2)\,(a_2x_1 + a_1a_5),$$

$$(a_1y_1 + a_3a_4)\,x_1^2 = (a_4y_1 + a_1a_3)\,(a_3y_1 + a_1a_4).$$

Eliminating y_1^2 and solving for y_1, we find

* Lamb, *Statics*, p. 71.

$$y_1 = \frac{a_3 a_4 \{2a_1 a_2 a_5 + (a_1^2 + a_2^2 + a_5^2) x_1 - x_1^3\}}{(a_1 x_1 + a_2 a_5)(x_1^2 - a_3^2 - a_4^2)} \quad \text{.................(14)}$$

which may be abbreviated to *

$$y_1 = (a_5, a_1, a_2 \; ; \; x_1)/2\mu_1 m_1 = X_1/2\mu_1 m_1, \quad \text{.................(15)}$$

where μ_1 and m_1 have been already defined, and

$$X_1 = (a_5, a_1, a_2 \; ; \; x_1) = 2a_1 a_2 a_5 + (a_1^2 + a_2^2 + a_5^2) x_1 - x_1^3. \quad \text{..........(16)}$$

Substituting from (15) in the first of equations (13) we obtain

$$X_1^2 = 4\lambda_1 \mu_1 \nu_1 m_1^2. \quad \text{.................................(17)}$$

This is the equation of the seventh degree satisfied by x_1, referred to in § 9. In full, it reads

$$\begin{aligned} &a_3^2 a_4^2 \{2a_1 a_2 a_5 + (a_1^2 + a_2^2 + a_5^2) x_1 - x_1^3\}^2 \\ &\quad = (a_5 x_1 + a_1 a_2)(a_1 x_1 + a_2 a_5)(a_2 x_1 + a_1 a_5)(x_1^2 - a_3^2 - a_4^2)^2, \quad \text{.................(18)} \end{aligned}$$

or, when expanded,

$$\begin{aligned} &a_1 a_2 a_5 x_1^7 + (a_1^2 a_2^2 + a_2^2 a_5^2 + a_1^2 a_5^2 - a_3^2 a_4^2) x_1^6 \\ &\quad + a_1 a_2 a_5 (a_1^2 + a_2^2 + a_5^2 - 2a_3^2 - 2a_4^2) x_1^5 \\ &\quad + \{a_1^2 a_2^2 a_5^2 - 2(a_3^2 + a_4^2)(a_1^2 a_2^2 + a_2^2 a_5^2 + a_1^2 a_5^2) + 2a_3^2 a_4^2 (a_1^2 + a_2^2 + a_5^2)\} x_1^4 \\ &\quad + a_1 a_2 a_5 \{a_3^4 + a_4^4 + 6a_3^2 a_4^2 - 2(a_3^2 + a_4^2)(a_1^2 + a_2^2 + a_5^2)\} x_1^3 \\ &\quad + \{(a_3^2 + a_4^2)^2 (a_1^2 a_2^2 + a_2^2 a_5^2 + a_1^2 a_5^2) - 2(a_3^2 + a_4^2) a_1^2 a_2^2 a_5^2 \\ &\qquad\qquad - a_3^2 a_4^2 (a_1^2 + a_2^2 + a_5^2)^2\} x_1^2 \\ &\quad + a_1 a_2 a_5 (a_1^2 + a_2^2 + a_5^2)(a_3^2 - a_4^2)^2 x_1 \\ &\quad + a_1^2 a_2^2 a_5^2 (a_3^2 - a_4^2)^2 = 0. \quad \text{.................(19)} \end{aligned}$$

11. To show that the other diagonals can be expressed as rational functions of x_1, we note first that λ_1, μ_1, ν_1 are linear functions of x_1, that m_1 is a quadratic in x_1, and that X_1 is a cubic . Next, by interchanging first a_1 and a_5 and then a_1 and a_2 in (14), we find $x_4 = X_1/2\lambda_1 m_1$ and $x_3 = X_1/2\nu_1 m_1$. Thus x_4, y_1, x_3, given by

$$x_4 = X_1/2\lambda_1 m_1, \quad y_1 = X_1/2\mu_1 m_1, \quad x_3 = X_1/2\nu_1 m_1, \quad \text{...............(20)}$$

are all rational in x_1. That the same is true of x_2, x_5, y_2, y_3, y_4, y_5 follows from (4·2), (3·2), (6·3), (3·3), (4·1), (6·1) respectively, provided that $x_1 \neq 0$.

Besides the diagonals, the cosines l_1, m_1, n_1, ... are also rational in x_1, since, by § 6, they can be expressed rationally in terms of diagonals. The same is evidently true of r^2, the square of the radius of the circumcircle, since

$$x_1^2 = 4r^2 \sin^2(\theta_3 + \theta_4) = 4r^2(1 - m_1^2) \; ; \quad \text{.....................(21)}$$

and it follows from the definition of S in (9) that the same is true of the product rS.

12. From (11) and (21) we have

$$r^2 = a_3^2 a_4^2 x_1^2 / \{(a_3 + a_4)^2 - x_1^2\} \{x_1^2 - (a_3 - a_4)^2\}, \quad \text{...............(22)}$$

from which we infer that in order that the corresponding circle may be real, it is necessary that x_1, besides being real, should satisfy the inequalities

$$| a_3 - a_4 | \leqslant | x_1 | \leqslant a_3 + a_4. \quad \text{.........................(23)}$$

Consequently, if we construct the equation of the seventh degree satisfied by x_1^2, then a necessary condition that all seven circles are real will be that all the roots of the new equation should lie between $(a_3 - a_4)^2$ and $(a_3 + a_4)^2$.

* It may be verified that both m_1 and X_1 vanish when x_1 is a diameter of the circumcircle.

It may be noted that equation (19) has at least three real roots if the lengths of the bars are such that a linkage can be constructed from them, i.e. if the length of the longest bar is less than the sum of the lengths of the other four bars. For, if the left-hand side of (19) be denoted by $f(x_1)$, then $f(-\infty)<0$, $f(0)>0$, $f(+\infty)>0$. It follows that the number of negative roots is odd, and that the number of positive roots is even. But there must be at least one positive root, corresponding to the convex cyclic configuration. Hence the number of real roots must be three, five, or seven.

13. The product rS can be simply expressed in various ways in terms of diagonals. Thus, in (9) we can put

$$a_1l_1+a_2l_2+a_3l_3+a_4l_4+a_5l_5 \equiv (a_1l_1+a_2l_2+a_5l_5+m_1x_1)+(a_3l_3+a_4l_4-m_1x_1),$$

and since the following two equations are identities when expressed in terms of the angles $\theta_1, \theta_2, \ldots$

$$2r^2(a_1l_1+a_2l_2+a_5l_5+m_1x_1)=x_3x_4y_1,$$

$$2r^2(a_3l_3+a_4l_4-m_1x_1) \qquad = a_3a_4x_1,$$

we deduce that

$$4rS=a_3a_4x_1+x_3x_4y_1. \quad\quad (24)$$

This is one of ten such formulae, each of which, divided by $4r$, expresses the area S as the sum of the area of a triangle and the area of a quadrilateral.

In a similar way, we can find fifteen formulae, each of which, divided by $4r$, expresses the area S as the sum of the areas of three triangles. Typical formulae of this kind are

$$\left.\begin{aligned} 4rS&=a_1x_2x_5+a_4a_5x_2+a_2a_3x_5,\\ 4rS&=a_1y_3y_4+a_2a_4y_3+a_3a_5y_4,\\ 4rS&=a_1x_1y_1+a_3a_4x_1+a_2a_5y_1.\end{aligned}\right\} \quad\quad (25)$$

The interest of these formulae, which are evident enough in the convex case, is that they hold good in the other cases as well.

14. In § 11 the proviso was made that $x_1 \neq 0$, which was required by the method of finding the last six diagonals in terms of x_1. The case $x_1=0$ occurs when $a_3=a_4$, and equation (19) then has two zero roots. To discuss this case analytically it would be necessary to begin by putting $a_3=a_4$ and $x_1=0$ in the equations of § 7. Geometrically, it is plain that, in this case, the corresponding two circles will be equal, each being the circumcircle of the triangle of sides a_1, a_2, a_5, and that each of the two corresponding cyclic configurations of the linkage will consist of this triangle inscribed in its own circumcircle, with the two equal links a_3, a_4 in coincidence along a chord of the circle. A necessary condition for these configurations to be real will therefore be that the diameter of the circumcircle of the triangle of sides a_1, a_2, a_5 should exceed the common length of the two links a_3, a_4.

Another special case that may be mentioned is that in which all the bars are equal in length. If we denote the common length by a, then equation (19) reduces to

$$(x_1^2-ax_1-a^2)(x_1+a)^3x_1^2=0.$$

The roots of the quadratic equation $x_1^2-ax_1-a^2=0$ are $x_1=2a \sin 54°$ and $x_1=-2a\sin 18°$, the first of which corresponds to the linkage in the shape of a regular pentagon, the second to the linkage in the shape of a pentagram, or regular five-pointed star. The other five roots correspond to an inscribed equilateral triangle, with three of the bars in coincidence along one side of the triangle in each case, the bars that coincide being $a_5a_1a_2$, $a_1a_2a_3$, $a_2a_3a_4$, $a_3a_4a_5$, $a_4a_5a_1$ respectively. In two of these cases, the bars a_3, a_4 coincide,

corresponding to the two roots $x_1=0$, $x_1=0$; the other three cases correspond to the roots $x_1=-a$, $x_1=-a$, $x_1=-a$.

15. The problem of finding the equation satisfied by r^2, the square of the radius of the circumcircle, and by S^2, the square of the area, of any cyclic polygon of given sides, was considered by Möbius.* In his paper, he distinguished between odd and even polygons, pointing out (i) that polygons with an odd number of sides have the property, used above, that the sum of the angles 2θ is an odd multiple of $360°$ when measured in one sense, but is an even multiple of $360°$ when measured in the other sense; (ii) that the cyclic configurations of polygons with an even number of sides fall into two classes: one class has the property that the sum of the angles 2θ is an even multiple of $360°$, while in the other class the sum of the angles is an odd multiple of $360°$, in whichever sense they are measured.

In the same paper Möbius found a simple formula for the degree of the equation satisfied by r^2 or S^2; he showed in particular that the degree for an odd polygon of $2m+1$ sides is

$$\frac{(2m+1)!}{2(m!)^2}-2^{2m-1}.$$

He did not, however, find the equations themselves, even for the case of $2m+1=5$. The object of the present paper has therefore been to show that the complete solution of the problem in this case can be made to depend upon an equation of the seventh degree which can be expressed in the simple form of (17).

The method used by Möbius was, in the case of the pentagon, to start from the equation

$$\sin(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)=0$$

and to make this rational in $\sin\theta_1$, $\sin\theta_2$, ... by deducing from it the equation

$$\sin(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)\Pi\sin(-\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)\,\Pi\sin(-\theta_1-\theta_2+\theta_3+\theta_4+\theta_5)=0,$$

in which the product indicated by the first Π consists of five factors, in each of which one angle is preceded by the minus sign, and the product indicated by the second Π consists of ten factors, in each of which two angles are preceded by the minus sign. When, in this rationalised equation, we substitute for $\sin\theta_1$, $\sin\theta_2$, ... by putting

$$2\sin\theta_1=a_1/r,\quad 2\sin\theta_2=a_2/r,\ \dots,\quad 2\sin\theta_5=a_5/r,$$

we obtain the equation satisfied by r^2, with coefficients which are symmetric functions of the sides. Möbius showed that this equation is of the seventh degree, but did not attempt to write out the full equation, which appears to be lengthy, the product of the seven roots alone, given by him, being equal to

$$a_1^6a_2^6a_3^6a_4^6a_5^6/(a_1+a_2+a_3+a_4+a_5)\Pi(-a_1+a_2+a_3+a_4+a_5)\,\Pi(-a_1-a_2+a_3+a_4+a_5),$$

where the denominator is the product of $1+5+10=16$ factors. This may be compared with the product of the seven values of x_1, which, by (19), is simply

$$-a_1a_2a_5(a_3^2-a_4^2)^2.$$

F. B.

* A. F. Möbius, "Ueber die Gleichungen, mittelst welcher aus den Seiten eines in einer Kreis zu beschreibenden Vielecks der Halbmesser des Kreises und die Fläche des Vielecks gefunden werden", *Crelle's Journal*, 3 (1828), 5–34.

A RESULT CONCERNING SEQUENCES OF INTEGERS.

By M. P. Drazin.

1. In a recent issue of the *American Mathematical Monthly*,* L. Moser proved the following result :

If an enthusiastic problemist proposes at least one problem every day, but never more than 730 in any calendar year (not even in a leap year), then, given any positive integer n, there is some set of consecutive days in which he proposes a total of exactly n problems.

In this note I obtain some generalizations of this result, using arguments similar to those of Moser.

Theorem 1. *Let $\{s_k\}$ be any strictly increasing sequence of (not necessarily positive) integers with the property that there exists a (strictly) increasing sequence of integers $\{k_p\}$ such that*

$$\overline{\lim_{p\to\infty}}\, s_{k_p}/k_p < 2. \tag{1}$$

Then, given any integer n, we can find integers i, j such that

$$s_i - s_j = n.$$

Proof. There will clearly be no loss of generality if we suppose that $n > 0$. We can find a positive constant δ, and an integer p_0, such that

$$s_{k_p}/k_p \leqslant 2 - \delta \text{ for every integer } p \geqslant p_0\,; \tag{2}$$

then, p_0, δ, being fixed, we can choose an integer $q \geqslant p_0$ such that

$$k_q > (n - s_1 + 1)/\delta \tag{3}$$

Consider now the two sets of *positive* integers

$$\{s_k - s_1 + 1\}, \quad \{s_k + n - s_1 + 1\} \quad (k = 1, 2, \ldots, k_q).$$

These sets together contain $2k_q$ integers, of which the greatest is

$$s_{k_q} + n - s_1 + 1 \leqslant k_q(2 - \delta) + n - s_1 + 1 < 2k_q$$

by (2), (3) ; hence, by the *Schubfach* principle (*i.e.* Dirichlet's " box argument "), since each of the sequences $s_k - s_1 + 1$, $s_k + n - s_1 + 1$ is *strictly* increasing, it follows that the two sets have a common element, say

$$s_i - s_1 + 1 = s_j + n - s_1 + 1$$

(with $i, j \leqslant k_q$), which immediately gives the required result.

2. It is easily verified that the hypothesis

$$\overline{\lim_{p\to\infty}}\, (s_{k_{p+1}} - s_{k_p})/(k_{p+1} - k_p) < 2$$

implies the hypothesis (1) of Theorem 1 ; in particular, taking $k_p = pm - d$, where m, d are given integers with $0 \leqslant d < m$, we see that the conclusion of the theorem will certainly hold if we replace the hypothesis (1) by

$$\operatorname*{\overline{bd}}_{p\geq 1}\, (s_{(p+1)m-d} - s_{pm-d})/m < 2.$$

Taking $m = 4.365 + 1 = 1461$, and letting s_k be the total number of problems concocted by Moser's " problemist " in the first k days of his creative activity (which we suppose to begin on the $(d+1)$th day of some given 4-year period),

* Vol. 57 (1950), p. 117.

then we have

$$s_{(p+1)m-d} - s_{pm-d} \leqslant 4 \,.\, 730 = 2m - 2\,;$$

consequently

$$\overline{\mathrm{bd}}_{p\geq 1}\, (s_{(p+1)m-d} - s_{pm-d})/m \leqslant 2 - (2/m) < 2,$$

and Moser's result follows immediately as a special case.

It is, of course, clear that Moser's result would be definitely false if we did not take leap years into account (or, analogously, if we allowed equality in (1) of Theorem 1), since a uniform output of two problems per day would then be admissible, with which *no* sequence of days could give an *odd* total of problems.

I am indebted to Mr. E. S. Barnes for a remark which has enabled me to put Theorem 1 into a form more general than my previous statement of it; I shall now obtain a generalization of Theorem 1 in a different direction.

THEOREM 2. *Let $\{s_k\}$ be any strictly increasing sequence of integers, with the property*

$$\underline{\lim}_{k\to\infty}\, s_k/k < \infty$$

(i.e. s_k/k does not tend to infinity with k).

Then, given any integer n, we can find integers i, j, and a positive integer $t \leqslant \underline{\lim}\, s_k/k$ such that

$$s_i - s_j = tn.$$

Proof. As before, we may suppose $n > 0$.

Choose an integer l such that

$$l \leqslant \underline{\lim}\, (s_k/k) < l+1\,;$$

then clearly $l \geqslant 1$, and, for a suitable $\delta > 0$, we can write

$$\underline{\lim}\, (s_k/k) = l+1-2\delta.$$

Then, by the definition of lower limit, we can certainly find a positive integer

$$m > (ln - s_1 + 1)/\delta$$

such that

$$s_m/m \leqslant \delta + \underline{\lim}\, s_k/k = l+1-\delta.$$

Consider now the $l+1$ sets

$$\{s_k - s_1 + 1\}, \quad \{s_k + n - s_1 + 1\}, \ldots, \quad \{s_k + ln - s_1 + 1\} \quad (k = 1, 2, \ldots, m).$$

These contain, in total, $m(l+1)$ positive integers, and the greatest of these is

$$s_m + ln - s_1 + 1 \leqslant m(l+1-\delta) + ln - s_1 + 1 < m(l+1),$$

so (as in the proof of Theorem 1) we can find integers i, j, and a positive integer $t \leqslant l$ such that $s_i - s_j = tn$; since $l \leqslant \underline{\lim}\, (s_k/k)$, the required result follows.

3. It might at first sight seem that, if the condition on s_k in Theorem 2 holds, *i.e.* if there is *some* increasing sequence of suffixes $\{k_p\}$ such that s_{k_p}/k_p is bounded, then this would necessarily hold for *all* increasing sequences $\{k_p\}$, so that the sequence s_k/k would itself be bounded.

This *is* in fact the case if we can find an increasing sequence $\{k_p\}$ of positive integers such that s_{k_p}/k_p, k_{p+1}/k_p are *both* bounded (say $k_{p+1} \leqslant \mu k_p$). For then, given any k, and choosing q so that $k_q \leqslant k < k_{q+1}$, we have

$$s_k/k < s_{k_{q+1}}/k_q = (k_{q+1}/k_q)\,.\,(s_{k_{q+1}}/k_{q+1}) \leqslant \mu\, \overline{\mathrm{bd}}\, (s_{k_p}/k_p),$$

the (finite) constant on the right being independent of k.

However, without this extra restriction that k_{p+1}/k_p be bounded, the condition on $\{k_p\}$ in Theorem 1 (or its natural extension along the lines of Theorem 2) *cannot* be extended to arbitrary sequences of suffixes; indeed, increasing integer sequences $\{s_k\}$ exist for which

$$\varliminf_{k\to\infty} s_k/k = 1, \quad \varlimsup_{k\to\infty} s_k/k = \infty,$$

i.e. for which s_k/k is unbounded, while also, given any $\epsilon > 0$, there is an increasing sequence $\{k_p\}$ such that s_{k_p}/k_p is bounded above by $1+\epsilon$ (and this is clearly as badly behaved an example as we can hope to find).

To demonstrate this, define

$$s_k = 2^{r^2+r} + k \text{ when } 2^{r^2} \leqslant k < 2^{(r+1)^2} \quad (r = 0, 1, 2, \dots)$$

(which is obviously a strictly increasing integer sequence); then, for $k = 2^{r^2}$,

$$s_k/k = 2^r + 1,$$

while, for $k = 2^{r^2+2r}$,

$$s_k/k = 2^{-r} + 1,$$

as required. M. P. D.

A NOTE ON SKEW-SYMMETRIC MATRICES.

By M. P. Drazin.

Given any skew-symmetric $n \times n$ matrix $\boldsymbol{A}$, we have

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det(\boldsymbol{A} - \lambda \boldsymbol{I})' = \det(-\boldsymbol{A} - \lambda \boldsymbol{I}) = (-1)^n \det(\boldsymbol{A} + \lambda \boldsymbol{I}),$$

whence we see that the non-zero eigenvalues of $\boldsymbol{A}$ can be arranged in pairs $\alpha, -\alpha$.* Since the set of n eigenvalues of $\boldsymbol{A}^2$ is precisely the set of the squares of the eigenvalues of $\boldsymbol{A}$, it follows that every non-zero eigenvalue of $\boldsymbol{A}^2$ occurs with *even* multiplicity, so that the characteristic function $\phi(\lambda) = \det(\boldsymbol{A}^2 - \lambda \boldsymbol{I})$ of $\boldsymbol{A}^2$, regarded as a polynomial in λ, is a *perfect square* if n is even, while, if n is odd, then we may write $\phi(\lambda) = \lambda\{f(\lambda)\}^2$ for a suitable polynomial $f(\lambda)$.

In this note I show that this result can be generalized so as to apply to the product of *any pair* of skew-symmetric matrices. We require the following

Lemma. *Given any even integer $n = 2m$, then there is a polynomial $p_m(a_{11}, a_{12}, \dots, a_{nn})$ with integer coefficients such that, if the (possibly complex) $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$ is skew-symmetric, then*

$$\det \boldsymbol{A} = \{p_m(a_{11}, a_{12}, \dots, a_{nn})\}^2.$$

We shall not give a proof of this result here, since it is well known, and is derived in most of the standard text-books on matrix theory.†

It will be convenient to use $\boldsymbol{a}$ to denote the row of n^2 variables $a_{11}, a_{12}, \dots, a_{nn}$ (and similarly for the elements of any given square matrix), so that we may write $p_m(a_{11}, a_{12}, \dots, a_{nn}) = p_m(\boldsymbol{a})$.

* If $\boldsymbol{A}$ is *real* then these are, in fact, conjugate pure imaginary quantities; however, this point is of no special interest in the present discussion.

† See, *e.g.*, A. C. Aitken, *Determinants and Matrices* (Edinburgh, 1942), 49, or W. L. Ferrar, *Algebra* (Oxford, 1941), 60.

THEOREM 1.* *Given any even integer n, then there is a polynomial $q(\lambda)=q(\boldsymbol{a}, \boldsymbol{b}, \lambda)$ in the $2n^2+1$ variables $\boldsymbol{a}, \boldsymbol{b}, \lambda$, with integer coefficients, such that if both the $n \times n$ matrices $\boldsymbol{A}=(a_{ij})$, $\boldsymbol{B}=(b_{ij})$ are skew-symmetric, then*

$$\det(\boldsymbol{AB}-\lambda \boldsymbol{I})=\{q(\lambda)\}^2.$$

Proof. Suppose first that each of $\boldsymbol{A}$, $\boldsymbol{B}$ is non-singular, and write $\boldsymbol{A}^{-1}=\boldsymbol{C}$, $\boldsymbol{B}^{-1}=\boldsymbol{D}$, $n=2m$.

Then plainly $\boldsymbol{C}$ is skew-symmetric, so that $(\boldsymbol{B}-\lambda\boldsymbol{C})'=-(\boldsymbol{B}-\lambda\boldsymbol{C})$, and therefore

$$\det(\boldsymbol{AB}-\lambda\boldsymbol{I})=\det(\boldsymbol{A}(\boldsymbol{B}-\lambda\boldsymbol{C}))=\det\boldsymbol{A}\,.\det(\boldsymbol{B}-\lambda\boldsymbol{C}),$$

where $\boldsymbol{A}$, $\boldsymbol{B}-\lambda\boldsymbol{C}$ are both skew-symmetric $n \times n$ matrices. Hence, by the Lemma, we have

$$\det(\boldsymbol{AB}-\lambda\boldsymbol{I})=\{p_m(\boldsymbol{a})p_m(\boldsymbol{b}-\lambda\boldsymbol{c})\}^2, \tag{1}$$

so we have now to show that, when $\boldsymbol{c}$ is expressed explicitly in terms of the elements of $\boldsymbol{A}$ (*i.e.* of $\boldsymbol{a}$), then $q(\boldsymbol{a}, \boldsymbol{b}, \lambda)=p_m(\boldsymbol{a})\,p_m(\boldsymbol{b}-\lambda\boldsymbol{c})$ (which is obviously a polynomial with respect to $\boldsymbol{b}$, λ) is also in fact a polynomial in the elements of $\boldsymbol{A}$, *i.e.* that the terms in $1/(\det\boldsymbol{A})$ arising from $\boldsymbol{c}$ in $p_m(\boldsymbol{b}-\lambda\boldsymbol{c})$ are " cancelled out " when multiplied by $p_m(\boldsymbol{a})$.

If this were not the case, then, since $\boldsymbol{c}$ is a *rational* functionof $\boldsymbol{a}$, we could write

$$q(\boldsymbol{a}, \boldsymbol{b}, \lambda)=\frac{q_1(\boldsymbol{a}, \boldsymbol{b}, \lambda)}{q_2(\boldsymbol{a})},$$

where $q_1(\boldsymbol{a}, \boldsymbol{b}, \lambda)$, $q_2(\boldsymbol{a})$ are *coprime* polynomials in their arguments, and $q_2(\boldsymbol{a})$ has positive degree ; then we should have

$$\{q_1(\boldsymbol{a}, \boldsymbol{b}, \lambda)\}^2=\{q_2(\boldsymbol{a})\}^2\det(\boldsymbol{AB}-\lambda\boldsymbol{I}),$$

so that any irreducible factor of $q_2(\boldsymbol{a})$ would divide $q_1(\boldsymbol{a}, \boldsymbol{b}, \lambda)$, contrary to our hypothesis that $q_1(\boldsymbol{a}, \boldsymbol{b}, \lambda)$, $q_2(\boldsymbol{a})$ are coprime.

Thus we see that $q(\boldsymbol{a}, \boldsymbol{b}, \lambda)$ *is* a polynomial of the required type, while also, by (1), we have

$$\det(\boldsymbol{AB}-\lambda\boldsymbol{I})=\{q(\boldsymbol{a}, \boldsymbol{b}, \lambda)\}^2$$

whenever $\boldsymbol{A}$, $\boldsymbol{B}$ are both non-singular.

The result for general (skew) $\boldsymbol{A}$, $\boldsymbol{B}$ now follows at once, since $\det(\boldsymbol{AB})$ is a polynomial in the elements of $\boldsymbol{A}$, $\boldsymbol{B}$, and is not identically zero.

This completes the proof of the theorem ; however, we can also prove that $q(\boldsymbol{a}, \boldsymbol{b}, \lambda)$ is a polynomial in another way. For, interchanging $\boldsymbol{A}$, $\boldsymbol{B}$ in (1), we have

$$\det(\boldsymbol{BA}-\lambda\boldsymbol{I})=\{p_m(\boldsymbol{b})p_m(\boldsymbol{a}-\lambda\boldsymbol{d})\}^2,$$

while also clearly

$$\det(\boldsymbol{BA}-\lambda\boldsymbol{I})=\det(\boldsymbol{BA}-\lambda\boldsymbol{I})'=\det(\boldsymbol{AB}-\lambda\boldsymbol{I})$$

(since $\boldsymbol{A}$, $\boldsymbol{B}$ are skew) ; it follows that

$$p_m(\boldsymbol{a})p_m(\boldsymbol{b}-\lambda\boldsymbol{c})\equiv \pm p_m(\boldsymbol{b})p_m(\boldsymbol{a}-\lambda\boldsymbol{d}), \tag{2}$$

* I have found, during the proof stage of this note, that this theorem is, essentially, contained in some work of N. Jacobson, *Bull. American Math. Soc.*, vol. 45 (1939), pp. 745–748. He showed that, if $\boldsymbol{R}$, $\boldsymbol{C}$ are given $n \times n$ matrices such that $\boldsymbol{RC}=\boldsymbol{C}'\boldsymbol{R}$, and if $\boldsymbol{R}$ is skew-symmetric and non-singular, then the characteristic function of $\boldsymbol{C}$ is of the form $\{g(\lambda)\}^2$, where $g(\lambda)$ is a suitable polynomial (and, indeed, $g(\boldsymbol{C})=0$) ; Theorem 1 follows (for non-singular $\boldsymbol{B}$) on taking $\boldsymbol{R}=\boldsymbol{B}$, $\boldsymbol{C}=\boldsymbol{AB}$. However, my proof immediately extends so as to apply to this more general case, and thus furnishes a simple alternative derivation of Jacobson's result.

for a suitable (fixed) determination of the ambiguous sign.*

Now, if we replace $\boldsymbol{c}$, $\boldsymbol{d}$ by their expressions in terms of the elements of $\boldsymbol{A}$, $\boldsymbol{B}$, then, since $\boldsymbol{c}$ depends only on $\boldsymbol{A}$, clearly the expression on the left of (2), regarded as a function of $\boldsymbol{a}$, $\boldsymbol{b}$, λ, is a polynomial in $\boldsymbol{b}$, λ, while similarly the term on the right is a polynomial in $\boldsymbol{a}$, λ, so the result follows as before.

THEOREM 2. *Given any odd integer n, then there is a polynomial $r(\lambda)=r(\boldsymbol{a}, \boldsymbol{b}, \lambda)$ in the $2n^2+1$ variables $\boldsymbol{a}$, $\boldsymbol{b}$, λ, with integer coefficients, such that, if both the $n\times n$ matrices $\boldsymbol{A}=(a_{ij})$, $\boldsymbol{B}=(b_{ij})$ are skew-symmetric, then*

$$\det(\boldsymbol{AB}-\lambda\boldsymbol{I}) = -\lambda\{r(\lambda)\}^2.$$

Proof. Given that $\boldsymbol{A}$, $\boldsymbol{B}$ are skew-symmetric, define $(n+1)\times(n+1)$ matrices

$$\boldsymbol{A}_1=\begin{pmatrix}\boldsymbol{A} & \mathbf{0}\\ \mathbf{0} & 0\end{pmatrix}, \quad \boldsymbol{B}=\begin{pmatrix}\boldsymbol{B} & \mathbf{0}\\ \mathbf{0} & 0\end{pmatrix}.$$

Then, by Theorem 1, we have

$$\det(\boldsymbol{A}_1\boldsymbol{B}_1-\lambda\boldsymbol{I}_{n+1}) = \{q(\boldsymbol{a}_1, \boldsymbol{b}_1, \lambda)\}^2,$$

where $\boldsymbol{I}_{n+1}$ denotes the unit matrix of order $n+1$; also

$$\boldsymbol{A}_1\boldsymbol{B}_1-\lambda\boldsymbol{I}_{n+1}=\begin{pmatrix}\boldsymbol{AB}-\lambda\boldsymbol{I}_n & \mathbf{0}\\ \mathbf{0} & -\lambda\end{pmatrix},$$

so that

$$\begin{aligned}-\lambda\det(\boldsymbol{AB}-\lambda\boldsymbol{I}_n) &= \det(\boldsymbol{A}_1\boldsymbol{B}_1-\lambda\boldsymbol{I}_{n+1})\\ &=\{q(\boldsymbol{a}_1, \boldsymbol{b}_1, \lambda)\}^2.\end{aligned}$$

Since λ divides the term on the left of this equation, clearly $q(\boldsymbol{a}_1, \boldsymbol{b}_1, \lambda)$ must be divisible by λ, while also $\boldsymbol{a}_1$, $\boldsymbol{b}_1$ depend only on $\boldsymbol{A}$, $\boldsymbol{B}$, so we may write

$$q(\boldsymbol{a}_1, \boldsymbol{b}_1, \lambda)=\lambda r(\boldsymbol{a}, \boldsymbol{b}, \lambda);$$

then $r(\boldsymbol{a}, \boldsymbol{b}, \lambda)$ clearly has all the required properties.

We conclude with some brief remarks on the above results; it will be sufficient to consider only the case in which $n=2m$ is even.

In the first place, if we regard $q(\boldsymbol{a}, \boldsymbol{b}, \lambda)$ as a polynomial in the single variable λ, then it is easy to see that we cannot improve the result of Theorem 1 by placing any further non-trivial restriction on $q(\lambda)$. For, given any polynomial $g(\lambda)$ of degree m with leading term $\pm\lambda^m$, we can find an $m\times m$ matrix $\boldsymbol{M}$ such that

$$\det(\boldsymbol{M}-\lambda\boldsymbol{I}) = \pm g(\lambda);$$

then, on taking

$$\boldsymbol{A}=\begin{pmatrix}\mathbf{0} & \boldsymbol{M}\\ -\boldsymbol{M}' & \mathbf{0}\end{pmatrix}, \quad \boldsymbol{B}=\begin{pmatrix}\mathbf{0} & -\boldsymbol{I}_m\\ \boldsymbol{I}_m & \mathbf{0}\end{pmatrix},$$

clearly $\boldsymbol{A}$, $\boldsymbol{B}$ are skew, while also

$$\det(\boldsymbol{AB}-\lambda\boldsymbol{I}) = \det\begin{pmatrix}\boldsymbol{M}-\lambda\boldsymbol{I}_m & \mathbf{0}\\ \mathbf{0} & \boldsymbol{M}'-\lambda\boldsymbol{I}_m\end{pmatrix} = \{g(\lambda)\}^2.$$

Thus *any* given polynomial $q(\lambda)=\pm\lambda^m+q_1\lambda^{m-1}+\ldots+q_m$ can arise in the way stated in Theorem 1, for suitable $\boldsymbol{A}$, $\boldsymbol{B}$.

If $\boldsymbol{B}=\boldsymbol{A}$, or if $\boldsymbol{A}$, $\boldsymbol{B}$ are *real* and one of $i\boldsymbol{A}$, $i\boldsymbol{B}$ is a *positive definite* hermitian matrix (it being understood in each case that each of $\boldsymbol{A}$, $\boldsymbol{B}$ is skew), then it is possible to show further that $\boldsymbol{AB}$ has all its elementary divisors *linear* (*i.e.* that there is a non-singular matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is *diagonal*); however, the above example serves to show that this is not generally true.

M. P. D.

* In fact, taking $\lambda=0$, we see at once that the upper sign must always hold; however, we do not need to use this.

POLYGONS INSCRIBED IN POLYGONS.

POLYGONS OF n SIDES AND OF $2n$ SIDES INSCRIBED WITHIN A GIVEN POLYGON OF n SIDES.

BY H. V. LOWRY.

IF any point P_1 is taken on the side A_2A_3 of the triangle $A_1A_2A_3$ and P_1P_2 is drawn parallel to A_2A_1 to meet A_3A_1 in P_2, P_2P_3 parallel to A_3A_2 to meet A_1A_2 in P_3, P_3Q_1 parallel to A_1A_3 to meet A_2A_3 in Q_1 and then Q_1Q_2, Q_2Q_3, Q_3R_1 are drawn parallel to P_1P_2, P_2P_3, P_3Q_1 in the same way, finishing with R_1 on A_2A_3, it is easily shown that R_1 coincides with P_1, as in Fig. 1.

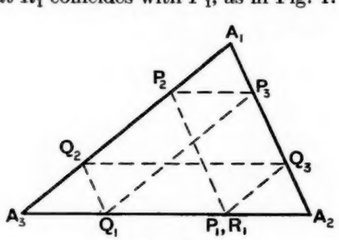

FIG. 1.

This suggested that it would be interesting to see if (i) a similar property held for any polygon, (ii) a similar property held for a triangle or a polygon when the angles of the polygon $P_1P_2P_3$ ... were any given angles, and Q_1Q_2, Q_2Q_3, ... were parallel to P_1P_2, P_2P_3, ...

The following investigation shows that in many cases a similar property does hold and some of these are quite interesting.

In order to make the investigation general we start with a polygon $A_1A_2 \ldots A_n$ of n sides and let P_{01}, P_{12}, ... $P_{n-1,n}$, $P_{n,1}$ be points on the sides A_nA_1, A_1A_2, ... $A_{n-1}A_n$, A_nA_1 respectively and Q_{01}, Q_{12}, ... Q_{n1} be points on the same sides such that Q_{01} is the same as P_{n1} and $Q_{r,\,r+1}$, $Q_{r+1,\,r+2}$ is parallel to

$$P_{r,\,r+1}P_{r+1,\,r+2}$$

for all r. We shall examine (i) under what conditions as to the position of P_{01} and the directions of the sides of the polygon formed by the P's the point Q_{n1} coincides with P_{01}; (ii) for what position of P_{01} the P polygon is closed, that is, P_{n1} coincides with P_{01} when the P polygon has to have given angles and a given orientation.

Let $P_{01}A_1 = x$, then (Fig. 2)

$$A_1P_{12} = x \sin A_1P_{01}P_{12}/\sin A_1P_{12}P_{01}$$
$$= \mu_1 x, \text{ say.}$$

In the same way,

$$A_2P_{23} = \mu_2 A_2P_{12}, \quad A_3P_{34} = \mu_3 A_3P_{23},$$

and so on, the values of the μ's depending on the size of the angles of the given polygon, on the size of the angles of the P polygon and on the orientation of the P polygon, which is fixed by the angle $A_1P_{01}P_{12}$, say. Hence, if the length of the side A_rA_{r+1} is $a_{r,\,r+1}$,

$$A_2P_{23} = \mu_2(a_{12} - \mu_1 x),$$
$$A_3P_{34} = \mu_3\{a_{23} - \mu_2(a_{12} - \mu_1 x)\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore,

$$\begin{aligned} A_1P_{n1} &= a_{n1} - \mu_n[a_{n-1,\,n} - \mu_{n-1}\{a_{n-2,\,n-1} - \dots\}] \\ &= a_{n1} - \mu_n a_{n-1,\,n} + \mu_n\mu_{n-1}a_{n-2,\,n-1} - \dots \\ &\qquad + (-1)^{n-1}\mu_n\mu_{n-1}\dots\mu_2 a_{12} + (-1)^n\mu_n\mu_{n-1}\dots\mu_1 x \\ &= b + \lambda x, \end{aligned}$$

where
$$b = a_{n1} - \mu_n a_{n-1,\,n} + \dots + (-1)^n\mu_n\mu_{n-1}\dots\mu_2 a_{12},$$
$$\lambda = (-1)^n\mu_n\mu_{n-1}\dots\mu_2\mu_1.$$

Thus the P polygon is closed only if $x = b + \lambda x$, that is, if $x = b/(1-\lambda)$. When $\lambda \neq 1$, this shows that, for given values of the μ's, that is, for a given shape and orientation of the P polygon, there is always one position and only one of P_{01} for which the P polygon is closed.

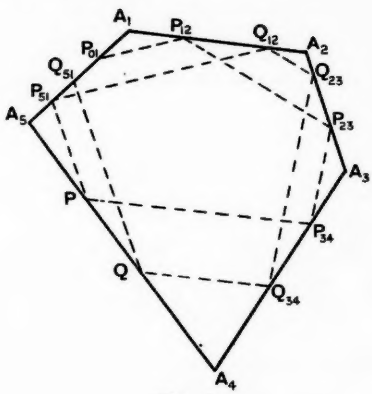

FIG. 2.

If $\lambda = 1$, the P polygon cannot be closed unless $b = 0$ and then it is closed whatever the value of x may be. Now, if the angles of the polygon are given, but not the orientation, b and λ are functions of one variable only, namely the angle $A_1P_{01}P_{12}$, which we will call θ, so that in general it is impossible to have $b = 0$ and $\lambda = 1$ at the same time and so we shall ignore this possibility hereafter.

We will now consider the closing of Q_{n1} on P_{01}. Since

$$A_1P_{n1} = b + \lambda x = b + \lambda A_1P_{01},$$
$$A_1Q_{n1} = b + \lambda A_1Q_{01} = b + \lambda A_1P_{n1} = b(1+\lambda) + \lambda^2 x.$$

Therefore Q_{n1} coincides with P_{01} if

$$x = b(1+\lambda) + \lambda^2 x,$$

whence, ignoring the possibility $b = 0$,

either $\quad \lambda \neq -1 \quad$ and $\quad x = b/(1-\lambda)$

or $\quad \lambda = -1 \quad$ for every value of x.

The first possibility is merely the same answer for the closing of the P polygon. This is natural because, if the P polygon is closed, the Q polygon will coincide with it.

If the P polygon has a given shape then λ is dependent on the one variable θ and hence λ can be -1 whenever the equation $\lambda = -1$ has real roots for θ.

Before proceeding to investigate cases in which the condition $\lambda = -1$ can be satisfied we note that, when $\lambda = -1$, if $A_1Q_{01} = x'$ for $A_1P_{01} = x$, and $A_1P_{01} = X$ when the polygon is closed,

$$x' = b - x, \qquad X = b - X.$$

Hence

$$X = \tfrac{1}{2}b = \tfrac{1}{2}(x + x').$$

Thus, whenever $\lambda = -1$ a double closed polygon can be drawn for any value of x and the value X of A_1P_{01} to make the single P polygon closed is midway between the values of A_1P_{01} and A_1P_{n1} in the double polygon.

Before proceeding to special cases in which $\lambda = -1$ we note that the directions given by the order of the letters in, for instance, A_1P_{12}, A_2P_{12}, etc. in Fig. 2 are the positive ones and, if P_{12} lies on A_1A_2 produced beyond A_2, then A_2P_{12} is negative. This means that if $\lambda = 1$ for a given set of μ's then $\lambda = -1$ if one side of the P polygon, say the last one $P_{n-1,\,n}P_{n1}$ is such that A_nP_{n1} is of the same length as for $\lambda = 1$ but in the opposite direction.

Special cases in which $\lambda = -1$, that is, in which the double polygon is closed for every value of X.

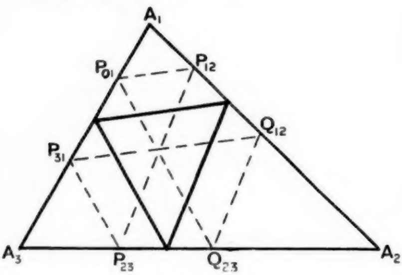

FIG. 3.

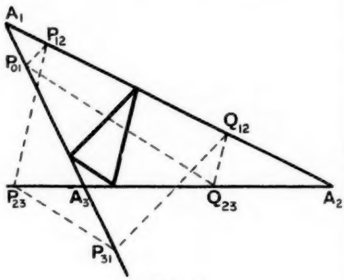

FIG. 4.

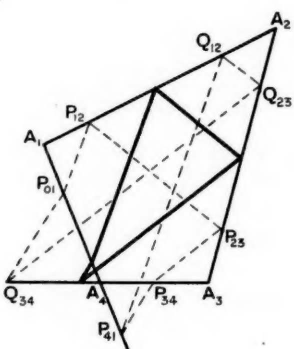

FIG. 5.

I. The most obvious case in which $\lambda = -1$ is when for n odd each of the μ's is equal to 1 and for n even one of the μ's (or an odd number of them) is -1

and the remainder of them equal to 1. In this case each triangle like $P_{01}A_1P_{12}$ is isosceles and the proof that the double polygon is closed is within the reach of the school certificate form. Each of Figs. 3, 4 shows a closed double polygon for a triangle by dotted lines and also the inscribed triangle obtained by taking $A_1P_{01}=\frac{1}{2}(x+x')$ by thick lines. Note that in Fig. 4, since A_3P_{23} is negative and $\mu=1$, A_2P_{31} has also to be taken negative.

Fig. 5 shows a closed double polygon for a quadrilateral by dotted lines and also the single polygon by a thick line. The number of sides being even in this case, one of the μ's has to be taken equal to -1; actually A_4P_{41}/A_4P_{34} is taken equal to -1 in the figure.

It is an interesting example in solution of simple equations to calculate the length of A_1P_{01} when the P polygon is closed, in terms of the lengths of the sides of the given polygon, for example in cases of the triangle and pentagon, say.

II. Let O be any point and let the angle $P_{r-1}P_rP_{r+1}$ be supplementary to the angle $A_{r-1}OA_r$. Let the angles OA_rA_{r-1} and OA_rA_{r+1} be A_r' and A_r'' respectively. Then it is easily shown that

$$\mu_1=\sin\theta/\sin(\theta+A_1'+A_1'')$$

and in general

$$\mu_r=\sin(\theta+A_1'-A_r')/\sin(\theta+A_1'+A_r''),$$

so that, if $\theta=-A_1'$, that is, $P_{01}P_{12}$ is parallel to OA_1,

$$\mu_1\mu_2\ldots\mu_n=(-1)^n\Pi(\sin A_r'/\sin A_r'').$$

But since

$$\sin A_r'/\sin A_{r-1}''=OA_r/OA_{r-1},$$

the above product is unity. Hence

$$\lambda=(-1)^n(-1)^n=1.$$

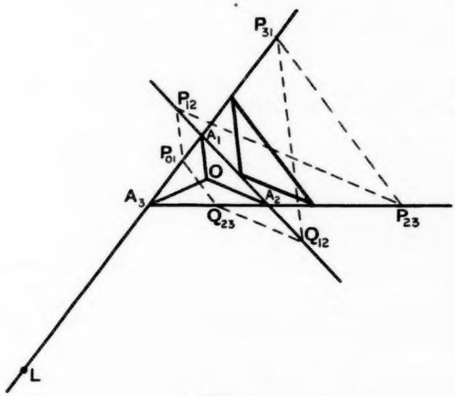

FIG. 6.

Therefore, if the sign of one of the μ's, say μ_r, is changed by drawing $A_rP_{r,\,r+1}$ away from A_{r+1} instead of towards it, keeping its length the same, then $\lambda=-1$ and the double polygon is closed for all values of x. In this case one of the angles of the P polygon is not supplementary to an angle at O. Fig. 6 shows

the construction in the case of a triangle but exactly the same holds for any polygon. The point L is that given by making the angle $P_{12}P_{23}L$ supplementary to A_3OA_1, and A_3P_{31} equal and opposite to A_3L. As before, the dotted lines form the double polygon and the thick lines form the triangle with its sides parallel to those of the double polygon.

III. If n is an odd number, $2k-1$, and the angles of the P polygon are made equal to the opposite angles of the given polygon, it is possible to put the equation $\lambda=-1$ into manageable form to find θ. The notation for this case is simplified if we give the number k to the side opposite A_k so that A_1A_2, being opposite A_{k+1}, is numbered $k+1$. If then we denote by α_{rs} the angle between the sides r and s and we use as variable for the orientation not θ but the angle ϕ which $P_{01}P_{12}$ makes with the side 1, that is, A_kA_{k+1}, we find that

$$\angle A_1P_{01}P_{12}=\alpha_{1,\,k}-\phi, \qquad \angle A_1P_{12}P_{01}=\alpha_{1,\,k+1}+\phi,$$

so that

$$\mu_1=\sin(\alpha_{1,k}-\phi)/\sin(\alpha_{1,k+1}+\phi).$$

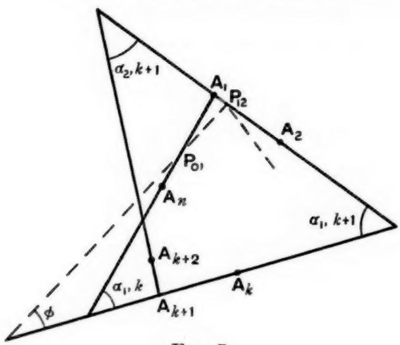

FIG. 7.

Fig. 7 shows that $A_{k+2}A_{k+1}A_k=180°-\alpha_{1,k+1}-\alpha_{2,k+1}$ and, since we are taking $\angle P_{01}P_{12}P_{23}$ equal to this angle, it follows that

$$\begin{aligned}\angle AP_{12}P_{23}&=180°-\alpha_{1,k+1}-\phi-(180°-\alpha_{1,k+1}-\alpha_{2,k+1})\\&=\alpha_{2,k+1}-\phi,\end{aligned}$$

and hence that

$$\mu_2=\sin(\alpha_{2,k+1}-\phi)/\sin(\alpha_{2,k+2}+\phi).$$

Continuing in this way we find that

$$\lambda=(-1)^{2k-1}\frac{\sin(\alpha_{1,k}-\phi)\sin(\alpha_{2,k+1}-\phi)\ldots\sin(\alpha_{2k-1,k-1}-\phi)}{\sin(\alpha_{1,k+1}+\phi)\sin(\alpha_{2,k+2}+\phi)\ldots\sin(\alpha_{2k-1,k}+\phi)}.$$

Moreover, since $\alpha_{sr}=\alpha_{rs}$, the α's in the denominator are the same as those in the numerator, but in a different order and consequently

$$\lambda=-\prod_{r=1}^{2k-1}\frac{\sin(\alpha_{r,k+r-1}-\phi)}{\sin(\alpha_{r,k+r-1}+\phi)}=-\Pi\frac{(t_r-t)}{(t_r+t)},$$

where $t=\tan\phi$ and $t_r=\tan\alpha_{r,k+r-1}$.

Thus $\lambda=-1$ always has the root $t=0$, that is, $\phi=0$, and hence if $P_{01}P_{12}$ is drawn parallel to A_kA_{k+1}, so that all the other sides of the P polygon are also parallel to the opposite sides of the given polygon, the double inscribed polygon is closed and of course one single inscribed polygon can be drawn by starting

with a point midway between P_{01} and P_n, in the double polygon. *In the case of the triangle this is the property that we started with*, and so we have shown that that property holds for any polygon with an odd number of sides. Fig. 8 shows such a double polygon inscribed in a pentagon and the corresponding single inscribed polygon.

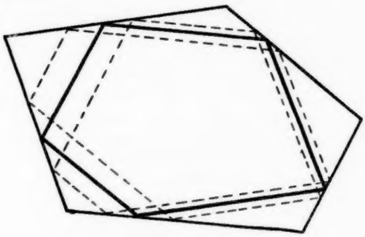

FIG. 8.

The equation for t shows that there may be other values of ϕ for which $\lambda = -1$. Taking a triangle, the equation is

$$(t_1 - t)(t_2 - t)(t_3 - t) = (t_1 + t)(t_2 + t)(t_3 + t),$$

that is, $$t = 0 \quad \text{or} \quad t^2 + \Sigma t_2 t_3 = 0.$$

Hence there is no other real value of t, or of ϕ, in the case of an acute-angled triangle, but in the case of an obtuse-angled triangle there are two equal and opposite real values of t, and hence of ϕ, for which $\lambda = -1$. For instance, for a triangle with angles 30°, 30°, 120°, we have

$$t_1 = t_2 = 1/\surd 3, \quad t_3 = -\surd 3, \text{ and hence } \Sigma t_2 t_3 = -5/3,$$

whence $\lambda = -1$ when $t = 0$ or $\pm\surd(5/3)$. Fig. 9 shows the two closed double polygons given by $t = \pm\surd(5/3)$.

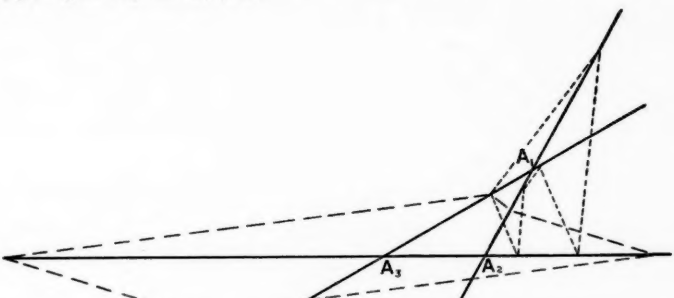

FIG. 9.

In conclusion we will consider what happens in III when neither the P polygon nor the Q polygon returns to P_{01}. Suppose that Z is the point reached on A_nA_1 after going round the sides of the given polygon p times, each time drawing lines parallel to the sides of the P polygon. Then

$$\begin{aligned} A_1Z &= b + \lambda[b + \lambda\{b + \dots\}] + \lambda^p x \\ &= b(1 + \lambda + \dots + \lambda^{p-1}) + \lambda^p x \\ &= b(1 - \lambda^p)/(1 - \lambda) + \lambda^p x. \end{aligned}$$

Thus Z coincides with P_{01} if

$$x = b(1 - \lambda^p)/(1 - \lambda) + \lambda^p x,$$

that is, if

$$(1 - \lambda^p)\{x - b/(1 - \lambda)\} = 0,$$

The equation has no further real roots than $\lambda = -1$ if p is even, and

$$x = b/(1 - \lambda)$$

for all values of p. Thus no new roots are found and so, if the polygon is not closed for $p = 1$ or $p = 2$, it cannot be closed for any other finite value of p. However, we note that if $|\lambda| < 1$,

$$A_1Z \to b/(1 - \lambda) \quad \text{as} \quad p \to \infty,$$

irrespective of the position of P_{01} and therefore the process tends in the limit to the single inscribed polygon of the given shape.

To examine just what happens in the case of a triangle we must find out the range of values of λ for which $|\lambda| < 1$. We have already seen that $\lambda = -1$ only when $t = 0$. The equation $\lambda = 1$ gives $t^2 \Sigma t_1 + t_1t_2t_3 = 0$. But in all cases,

$$\sum_{r=1}^{2k-1} \alpha_{r,k+r-1} = 180°,$$

and so in the case of the triangle, $\Sigma t_1 - t_1t_2t_3 = 0$. Consequently, the equation $\lambda = 1$ has no real roots. Consideration of the graph of λ then shows that $|\lambda| < 1$ only if t is positive that is, if $0 < \phi < 90°$. The process converges within this range of values of ϕ, but diverges outside this range. The convergence is shown in Fig. 10. The limiting triangle is, of course, the triangle inscribed in the triangle $A_1A_2A_3$ similar to $A_1A_2A_3$ with an orientation decided by the angle θ.

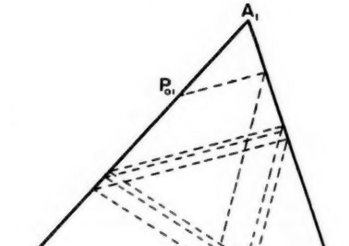

FIG. 10.

If the triangle is obtuse and $t_1 < t_2 < -t_3$, consideration of the graph of λ shows that, if $\lambda = -1$ when $t = \pm \tan\beta$, then $|\lambda| < 1$ either if $0 < \phi < \beta$ or $-\frac{1}{2}\pi < \phi < -\beta$ and the process converges if ϕ lies within either of these ranges.

H. V. L.

" DELTAHEDRA ".

BY H. MARTYN CUNDY.

THE substance of this note arose out of an illustration in the *Eighteenth Yearbook* of the (American) National Council of Teachers of Mathematics. On p. 299 of this work, which is full of many good things, there appears a diagram of a number of compound polyhedra. One of them is an icosahedron surmounted by an octahedron on every face, producing a solid which is simple and singly-connected (not counting the faces in common with the icosahedron itself). Since the dihedral angle of the icosahedron is 138° 11′ and that of the octahedron is 109° 28′, the total angle at each edge of the icosahedron is 357° 7′, and the solid thus has narrow fissures between each pair of octahedra. A slight deformation would close these fissures and produce a solid which consists entirely of equilateral triangular faces, which is neither regular nor Archimedean, but can be thought of as an Archimedean icosidodecahedron with pentagonal pyramids indented in its pentagonal faces.

This suggests a consideration of the class of solids whose faces are equilateral triangles, for which I suggest the name " deltahedra ". They have the great advantage that they are very easily made in the form of cardboard models, since their nets are pieces of the plane tessellation of equilateral triangles which is extremely easy to draw, and can even be obtained in printed form. Children who would find difficulty in constructing any of the more elaborate Archimedean solids can make deltahedra with ease. Every boy of a set can make an octahedron in a few minutes, and if the teacher meanwhile makes the icosahedron, the solid with which we opened this note can be made in a very short time.

The class of deltahedra is obviously infinite. It includes, for example, all the rotating rings of tetrahedra, whose construction is described in Rouse Ball, *Mathematical Recreations and Essays* (11th Revised Edition), p. 153. The class of *convex* deltahedra is, however, finite, but it is not confined to the three regular specimens. It has recently been shown by H. Freudenthal and B. L. van der Waerden (*Simon Stevin*, 25 (1947), pp. 115–21) that there are in fact eight such solids. My attention was kindly drawn to this paper by Prof. Coxeter of Toronto. The results are outlined here.

Euler's theorem for a polyhedron formed by triangles leads to the equation $3 \cdot v_3 + 2 \cdot v_4 + v_5 - v_7 - 2 \cdot v_8 - \ldots = 12$, where v_r is the number of vertices at which r triangles meet. A convex deltahedron must have $v_6 = v_7 = \ldots = 0$.

If v_3 is not zero, it is easy to see (by beginning to construct a deltahedron with a trihedral vertex) that we can only have the tetrahedron ($v_3 = 4$) and the triangular bipyramid ($v_3 = 2$, $v_3 = 3$). Otherwise $v_3 = 0$, and therefore

$$2 \cdot v_4 + v_5 = 12.$$

This equation has the following seven solutions :

Reference Letter	v_4	v_5	*No. of faces*	*Description*
[3, 5]	0	12	20	Icosahedron
—	1	10	—	Impossible
A	2	8	16	Square antiprism plus two square pyramids
B	3	6	14	Square antiprism plus one square pyramid plus one wedge
C	4	4	12	Square antiprism plus two wedges
D	5	2	10	Pentagonal bipyramid
[3, 4]	6	0	8	Octahedron

The " wedges " are each formed by two triangles whose free edges form the sides of a skew quadrilateral into which the square is distorted. In both solids the wedges are extremely flat. Solid C, named by Coxeter the " Siamese Dodecahedron ", can be thought of alternatively as consisting of two distorted pentagonal pyramids, joined together by three of their base edges, with a " wedge " occupying the skew quadrilateral formed by the four remaining edges. This solid has the symmetry of a tetragonal disphenoid : *i.e.* it is unaltered by reflection in either of two perpendicular planes, by rotation through 180° about their line of intersection, and by reflection in a plane perpendicular to them both followed by a rotation through 90°. Solid A has almost the same symmetry, but a tetragonal axis now joins two vertices instead of a diagonal axis joining the mid-points of opposite sides, and the angle of the rotatory-reflection is 45°. Solid B is more easily visualized as a triangular Archimedean prism with its square faces surmounted by square pyramids. Its symmetry is therefore trigonal : it has a plane of symmetry, and is unaltered by rotation through 120° about an axis perpendicular to this plane. Solid D, of course, has a plane of symmetry and a pentagonal axis perpendicular to it. Sketches of A, B, and C are appended. (Figs. 1, 2, 3.)

If we turn now to consider non-convex deltahedra, there are at least two ways in which we could attempt to classify them : either by the number of

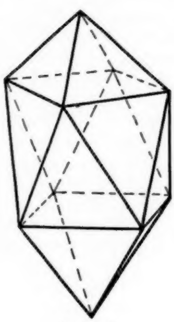

Fig. 1.

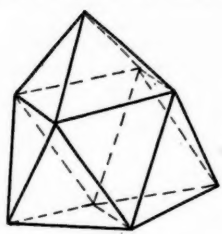

Fig. 2.

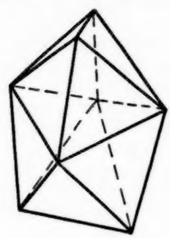

Fig. 3.

vertices, or by the number of different " types " of vertex. A set of vertices will be said to be of the same " type " if there is a subgroup of the symmetry group of the solid transitive on the set. It follows that the same number of faces meet at each vertex of a given type. To pursue this second method of classification first, there is only one deltahedron that is not convex with vertices of a single " type ", viz. the Great Icosahedron. I have been able to find seventeen non-convex deltahedra with two types of vertex. These are listed in the accompanying Table. Some of them are of interest in themselves. For example, No. 1 is the net of the regular four-dimensional simplex or *pentatope* ; No. 9 is one of the 59 Icosahedra (Ef_1g_1 : Coxeter, DuVal, Flather and Petrie, *The Fifty-Nine Icosahedra*, University of Toronto Studies (Mathematical Series), 6 (1938), Plate IX). Number 10 can be considered to have triangular and hexagonal faces, and is then a stellated " Archimedean polyhedron " with the hexagons in diametral planes through the centre. If the pyramids are " everted " in this case the result is an octahedron with its faces divided into four equilateral triangles : similarly if the pyramids of No. 13 are everted the result is an icosahedron with divided faces.

TABLE OF "DELTAHEDRA" WITH TWO KINDS OF VERTEX.

Method of Construction	Number of faces at each type of vertex		Total F	Total V	Total E	
1. Tetrahedra on faces of tetrahedron - - -	3	6	12	8	18	
2. Octahedra on faces of tetrahedron - - -	4	9	28	16	42	
3. Octrahedra on faces of octahedron - - -	4	12	56	30	84	
4. Tetrahedra on faces of icosahedron - - -	3	10	60	32	90	
5. Octahedra on faces of icosahedron - - -	4	15	140	72	210	
6. Tetrahedra on faces of octahedron - - - -	3	8	24	14	36	(Stella Octangula)
7. Pyramids on faces of cube (outwards) - - -	4	6	24	14	36	
8. Pyramids on faces of dodecahedron (outwards) -	5	6	60	32	90	
9. Pyramids on faces of dodecahedron (inwards) - -	5	6	60	32	90	
10. Pyramids on square faces of cuboctahedron (inwards)	4	6	32	18	48	
11. Pyramids on square faces of rhombicuboctahedron (outwards) - - -	4	7	80	42	120	
12. Pyramids on square faces of rhombicuboctahedron (inwards) - - -	4	7	80	42	120	
13. Pyramids on pentagonal faces of icosidodecahedron (inwards) - -	5	6	80	42	120	
14. Pyramids on square faces of snub cube (outwards) -	4	6	56	30	84	
15. Pyramids on square faces of snub cube (inwards) -	4	6	56	30	84	
16. Pyramids on pentagonal faces of snub dodecahedron (outwards) - -	5	6	140	72	210	
17. Pyramids on pentagonal faces of snub dodecahedron (inwards) - -	5	6	140	72	210	

If, in the case of No. 7, the pyramids are described inwards, they overlap and a peculiar re-entrant polyhedron results. The same will happen in a number of other cases; the table includes only those solids in which the triangles are totally on the outside.

The other method of classification follows the lines laid down by Brückner (*Vielecke und Vielfläche*, Tafel II and following) for trigonal polyhedra. The figures given in Brückner are for "general" polyhedra with three edges meeting at every vertex, but the dual polyhedra with triangular faces are easily obtained. It remains to test the possibility of constructing each solid with all its faces equilateral. It will be found that all the trigonal polyhedra

with 4, 5, 6, 7, or 8 vertices give rise to deltahedra. With 4 vertices we have the regular tetrahedron ; with 5 the triangular bipyramid. Six vertices lead either to the regular octahedron, or to the solid formed by adding two tetrahedra to the faces of a tetrahedron. The solids with 7 vertices include D, a tetrahedron on the face of an octahedron, and various ways of assembling four tetrahedra. With 8 vertices a partial exception arises in that the figure formed by joining together by their edges two regular hexagons, each divided into six triangles, is neither a " solid " nor rigid. All the others give genuine deltahedra, including C, and a number of solids formed by adding tetrahedra to one another or to the faces of an octahedron or a bipyramid D. Only one of these, No. 1 in the Table, has the symmetry of a regular solid.

When we try to do the same thing for 9 or more vertices we begin to encounter " exceptions " arising from the fact that the solids will be interpenetrating ; *e.g.* the case of six tetrahedra with a common edge. At the same time the number of possibilities is vastly increased and most of them have no interest, being merely solids which can be constructed by sticking together tetrahedra and octahedra in various ways. The requirement of symmetry of a regular solid does not limit the possibilities very greatly. For example, unlimited deltahedra can be constructed by adding equal stacks of octahedra to the faces of a basic icosahedron. In fact, of the making of deltahedra there is no end! H. MARTYN CUNDY.

1712. Of my first teacher at Dalry, Ayrshire, I have few personal recollections. He was a kindly man and an enthusiastic mathematician. His knowledge was probably as great as his enthusiasm, but of this I cannot speak definitely. It is told of him that, when paying his addresses to the worthy woman whom he subsequently made his wife, having exhausted the expressions of endearment that usually accompany courtship, he proceeded to exhibit to his sweetheart the beauty and difficulty of the forty-seventh proposition of the first book of Euclid.—Dr. John Kerr, *Leaves from an Inspector's Log Book*, quoted in *Scottish Educational Journal*. [Per Mr. R. W. Brownlee.]

1713. A translation from the Catechetical Lectures of S. Cyril of Jerusalem (" Proinde, numero aliquo rotundo posito, medio quarti seculi anno 348-350 Catacheses habitae sunt ") :

Sixty-nine week-years make 483 years. He [Daniel] says that, 483 years having elapsed since the building of Jerusalem and the reigning powers failing, a foreign king would come under whom Christ would be born. Now Darius built Jerusalem in the 6th year of his reign, but in the first year of the 66th Olympias. That period is called an Olympias which the difference, each fourth year, makes, on account of the day which it mounts to during the sun's four-year course, there being 3 hours left over each year. Herod obtained the kingdom in the 4th year of the 186th Olympias. The difference between the 66th and the 186th Olympias is 120 Olympiases plus. And the 120 Olympiases make 480 years. Those other 3 years (to complete the necessary number of weeks) are found in that interval between the first and fourth years. You now have proof of the Scripture saying : " . . . seven weeks and threescore and two weeks " ; although some different opinions are not wanting concerning the prediction in Daniel of the year-weeks.—12th lecture, sec. 19, *Cyrilli Catecheses Illuminandorum*, by G. C. Reischl, Monaci, 1848. [Per the Most Rev. the Archbishop of Yukon.]

A SIMPLY CONSTRUCTED ADDING MACHINE.

BY G. H. JOWETT.

STATISTICS is now obtaining a foothold in the curricula of some sixth forms in schools, and with its introduction as an option into parts of the General Certificate of Education, it is likely to be taught more in the future. One of the great difficulties which is being encountered is the amount of computation necessary in even the simplest applications of the science, and computation unfortunately of a type which is not greatly facilitated by a slide rule or table of logarithms. The bulk of the work, even in advanced applications, very frequently consists of (or may readily be reduced to) the summation of accurate squares and products of two-figure numbers such as form the raw material for applications of correlation, regression, and so forth.

The simply-constructed adding machine whose design is described here can be used in conjunction with a table of products (1) to facilitate computation of this kind. It may be laid out with a typewriter, and even duplicated if several are required; it requires only stiff paper (*e.g.* a foolscap manilla folder), thin card, (about 1 mm. thick), and a few wire staples as raw materials. Facility in its use comes quickly with practice, and requires no more mental effort, once the habit of use has been grasped, than any other more ambitious adding machine. Also, it is as durable as the cardboard slide rules which were cheap and common before the war.

Fig. 1 shows the layout of the body of the machine. This is made of manilla. Slits at positions (i), (ii), (iii), (v), are readily cut with a $\frac{1}{2}''$ chisel, and the window at position (iv) with a $\frac{1}{4}''$ chisel; alternatively, a penknife may be used.

(i)	——		——		——		——		——		——		——	
(ii)	——		——		——		——		——		——		——	
	—	0	—	0	—	0	—	0	—	0	—	0	—	0
	9—	1	9—	1	9—	1	9—	1	9—	1	9—	1	9—	1
	8—	2	8—	2	8—	2	8—	2	8—	2	8—	2	8—	2
	7—	3	7—	3	7—	3	7—	3	7—	3	7—	3	7—	3
	6—	4	6—	4	6—	4	6—	4	6—	4	6—	4	6—	4
	5—	5	5—	5	5—	5	5—	5	5—	5	5—	5	5—	5
	4—	6	4—	6	4—	6	4—	6	4—	6	4—	6	4—	6
	3—	7	3—	7	3—	7	3—	7	3—	7	3—	7	3—	7
	2—	8	2—	8	2—	8	2—	8	2—	8	2—	8	2—	8
	1—	9	1—	9	1—	9	1—	9	1—	9	1—	9	1—	9
	0—	-	0—	-	0—	-	0—	-	0—	-	0—	-	0—	-
(iii)	——		——		——		——		——		——		——	
(iv)	□		□		□		□		□		□		□	
(v)	——		——		——		——		——		——		——	

FIG. 1. Layout of Body of Machine.

Fig. 2*a* shows a typical slide, which is made of a piece of thin card $10\frac{1}{2}'' \times \frac{1}{2}''$, before insertion into the body of the machine. Card of this thickness is difficult to use in a typewriter, so the markings are best done by hand unless a large number of slides is required, when the slides may be laid out on a stencil (Fig. 2*b*) and duplicated. The slide is inserted into the face of the machine as shown in Fig. 2*c*; the ends are bent over as shown, and fastened into place with staples. The body of the machine, with all slides inserted, may

then be mounted on another piece of card, or better, pinned to a suitable piece of wood. When the slides have been manipulated a little so that they slide quite freely, the machine is ready for use.

FIG. 2a.
Specimen Slide.

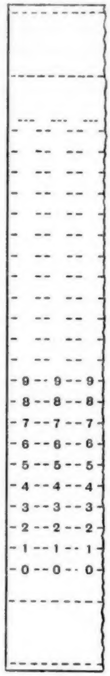

FIG. 2b.
Typewritten Multiple Slide Layout.

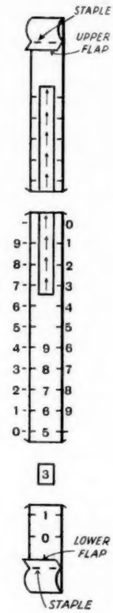

FIG. 2c.
Slide Inserted and End-flaps Made.

Addition and Multiplication.

To add, the right hand alone is used, the left being reserved for following the figures to be added, or for manipulating the multiplication tables.

Two types of adding movement must be cultivated :

(i) (no carry over is necessary). The index finger tip is placed on the slide opposite the digit (marked on the body to the left of the slide) which is to be added, the thumb against the lower flap. The latter is then pushed until the mark on the slide where the index finger rests is opposite zero.

(ii) (used when carry over is necessary). The tip of the thumb is placed on the slide opposite the number to be added, the index finger against the upper flap. The latter is then pushed until the thumb is opposite the short line just above the figure nine, and carry over is effected by pushing the adjacent left slide just one place downwards with the thumb ; during this last movement, the index finger is kept stationary to avoid losing the place.

The digits of numbers are equally well added from left to right or from right to left. The total appears in the little windows, which together form the register of the machine. Movement (ii) is required when the digit to be added lies against the upper part of the slide, which is marked with arrows. The selection and use of these movements becomes automatic with a little practice.

In the case of a double carry over, after the first carry over the top of the slide which has taken the one becomes visible, and the window becomes empty. It is then necessary to add zero by method (ii).

These movements are not as complicated as they sound, and a moderate adding speed is attained after about an hour's practice.

Example. Calculate Σxy from the following data :

$$x: \quad 21 \quad 75 \quad 91 \quad 37 \text{ etc.}$$
$$y: \quad 34 \quad 18 \quad 04 \quad 25 \text{ etc.}$$

The product $21 \times 34 = 0714$ is obtained from the multiplication table and added into the machine ; next the product $75 \times 18 = 1350$ is found from the table and added into the machine, giving a cumulative total of 2064 ; and so on. It is unnecessary to write the individual products down.

Example. Calculate $\cdot 3477 \times \cdot 7265$ to six places of decimals.

From Tables	*Added into Machine*	*Cumulative Total*
$34 \times 72 = 2448$	2448	244800
$34 \times 65 = 2210$	221	247010
$77 \times 72 = 5544$	5544	252554
$77 \times 65 = 5005$	50	252604 *Ans.*

Subtraction.

This operation is less commonly required in statistics. It is, however, readily carried out on the machine with the *left* hand using movements which are the reverse of the adding movements, and using the figures on the right of the slide. Carry over is conveniently effected by the third finger of the left hand. It is advisable to use different hands for addition and subtraction, since this avoids mistakes due to confusion of movements.

Division is probably best effected with the aid of a table of reciprocals.

The machine is a form of abacus, and many other machines based on this principle may be seen in the Science Museum. These are usually stylus-operated, and require the use of both hands. A small stylus-operated machine is on the market at present, and costs just less than £1 ; this is, of course, much more durable than the machine here described. G. H. J.

REFERENCES.

(1) Tables of Products.

To 999×999 : Crelle. (O. Seeliger.) *Calculating Tables.* Berlin, de Gruyter.

To 9999×999 : Peters, J. *New Calculating Tables.* Berlin, de Gruyter.

To 99×99 : Unwin, W. C. *Short Logarithmic and Other Tables.* London, Spoor.

A set of duplicated 99×99 tables has been prepared at the University of Sheffield for use with large classes in combination with machines of this type.

Machines and cards will be on view at the meeting of the Mathematical Association at Sheffield in April 1953.

THE ROOTS OF CERTAIN DETERMINANTAL EQUATIONS.

BY A. TALBOT.

IN her paper (*Gazette*, May 1950, p. 94) on the roots of two types of determinantal equation, M. J. Moore obtained various results by elementary but somewhat complicated methods . In this note, these results are obtained by a simpler method, which moreover helps to exhibit their real significance.

The essential facts are expressed by the following theorem :

THEOREM I. *If a rational function of x has poles and zeros (including a possible pole or zero at infinity) which are all real, simple and strictly alternate, then its partial fraction expansion*

$$k_\infty x + c + \sum_{r=1}^{\mu} k_r/(\gamma_r - x) \quad \text{.............................(1)}$$

has coefficients k_r (including k_∞ if non-zero) all of one sign, while its reciprocal has expansion coefficients all of the opposite sign.

Conversely, any such sum having coefficients all of the same sign is equal to a rational function of the above kind.

Both parts are immediately obvious from graphical considerations. An algebraical proof may also be readily given. Thus, to prove the first part, if a rational function has its poles and zeros real, simple and strictly alternate (for a definition of this, see Moore's paper, p. 94) then either the function or its reciprocal must be expressible as

$$\phi = A \prod_{t=1}^{m} (\beta_t - x) / \prod_{s=1}^{n} (\alpha_s - x), \quad \text{.........................(2)}$$

where $n = m$ or $m+1$, and

$$\alpha_1 < \beta_1 < \alpha_2 < \ldots < \beta_m (< \alpha_{m+1}). \quad \text{.........................(3)}$$

For $p \leqslant m$, the partial-fraction term $g_p/(\alpha_p - x)$ has coefficient

$$g_p = A \,.\, \prod_{t=1}^{m} (\beta_t - \alpha_p) / \prod_{\substack{s=1 \\ s \neq p}}^{n}{}' (\alpha_s - \alpha_p).$$

By (3) this has the sign of $A(-)^{p-1}/(-)^{p-1} = +A$. Similarly, for $q \leqslant m$, the term $h_q/(\beta_q - x)$ in the expansion of $1/\phi$ has coefficient

$$h_q = A \,.\, \prod_{s=1}^{n} (\alpha_s - \beta_q) / \prod_{\substack{t=1 \\ t \neq q}}^{m}{}' (\beta_t - \beta_q),$$

which has the sign of $A(-)^q/(-)^{q-1} = -A$. Also, if $n = m+1$, the term corresponding to the pole at infinity is $h_\infty x$, where $h_\infty = A(-)^n/(-)^m = -A$.

To summarise, all expansion coefficients for ϕ have the sign of A, for $1/\phi$ that of $-A$. The sign of the constant term, c, in (1) is not determined in this way. But if $m = n$, c has the sign of A for both ϕ and $1/\phi$.

To prove the converse result, let a rational function $\psi(x)$ have the expansion (1) in which $k_r > 0$, $k^\infty \geqslant 0$, and

$$-\infty < \gamma_1 < \gamma_2 < \ldots < \gamma_\mu < +\infty .$$

If ϵ is small and positive,

$$\psi(\gamma_r + \epsilon) \simeq -k_r/\epsilon \leqslant 0, \quad \psi(\gamma_{r+1} - \epsilon) \simeq k_{r+1}/\epsilon \geqslant 0,$$

so that there is an *odd* number of zeros of ψ in each of the intervals (γ_r, γ_{r+1}) $(r = 1, 2, \ldots, \mu - 1)$, and also in $(-\infty, \gamma_1)$ and $(\gamma_\mu, +\infty)$ if $k_\infty > 0$. If $k_\infty = 0$, the total number of zeros, that is, the degree of the numerator of ψ, is $\mu - 1$ if $c = 0$.

and μ if $c \neq 0$. Thus there must be just one real zero in each of the $(\mu - 1)$ intervals (γ_r, γ_{r+1}) and, if $c \neq 0$, one more in either $(-\infty, \gamma_1)$ or $(\gamma_\mu, +\infty)$. If $k_\infty > 0$, there are $(\mu + 1)$ zeros, that is, just one in each of the intervals $(-\infty, \gamma_1), (\gamma_1, \gamma_2), \ldots, (\gamma_\mu, +\infty)$. In all cases, the strict alternation of poles and zeros follows, and of course the zeros are all real and simple.

With the help of this theorem, it is a simple matter to discuss the determinant* Δ_n on p. 96. For, clearly,

$$\Delta_{n+1} = (x + b_{n+1}\Delta_n) - a_{n+1}c_n\Delta_{n-1},$$

so that

$$\Delta_{n+1}/\Delta_n = x + b_{n+1} - a_{n+1}c_n/(\Delta_n/\Delta_{n-1}).$$

Thus if Δ_n/Δ_{n-1} has strictly alternating real poles and zeros and an expansion with positive coefficients, so has $-a_{n+1}c_n/(\Delta_n/\Delta_{n+1})$, by the theorem (if $a_{n+1}c_n > 0$), and hence so has Δ_{n+1}/Δ_n. But

$$\Delta_2/\Delta_1 = x + b_2 - a_2c_1/(b_1 + x).$$

Hence by induction Δ_{n+1}/Δ_n has strictly alternating poles and zeros, and in particular Δ_m has m distinct real roots, for all m.

(A) Obviously, if $a_{n+1}c_n = 0$, then the roots of Δ_n are shared by Δ_{n+1}, $\Delta_{n+2}, \ldots$, but apart from these the zeros and poles of Δ_{n+1}/Δ_n, $\Delta_{n+2}/\Delta_{n+1}, \ldots$ are strictly alternate, that is, the remaining roots of $\Delta_{n+1}, \Delta_{n+2}, \ldots$ are real and distinct (but may coincide with the roots of Δ_n).

(B) In the proof of the converse part of the theorem, it is clear that if we know that $\gamma_\mu < 0$, and $\psi(0) > 0$, then since

$$\psi(\gamma_\mu + \epsilon) \simeq -k_\mu/\epsilon \ll 0,$$

ψ must have a zero in $(\gamma_\mu, 0)$ and so all zeros are negative. Thus taking ψ as Δ_{n+1}/Δ_n, if all roots of Δ_n are negative, and $\Delta_m(0) > 0$ for all m, it follows that all roots of Δ_{n+1} are negative.

To discuss the determinant Δ_n on p. 94, we need the following modified version of theorem I:

THEOREM II. *If an odd rational function $f(\lambda)$ of a complex variable $\lambda = \sigma + i\omega$ has poles and zeros which are simple and strictly alternate on the imaginary axis, then the partial-fraction expansions of the form*

$$c\lambda + \sum_{r=1}^{\mu} \frac{k_r\lambda}{\omega_r^2 + \lambda^2} \quad \ldots\ldots(4)$$

with $0 \leq \omega_1^2 < \omega_2^2 < \ldots < \omega_\mu^2$

for the function and its reciprocal, have coefficients k_r, and c if non-zero, all of the same sign.

Conversely, any such expansion represents an odd rational function of the above kind.

This follows easily from Theorem I. In the first part, if we write $x = -\lambda^2$, it is clear that f/λ and $1/\lambda f$ both have degree 0 or -1 in x with real alternating poles and zeros, the lowest being a pole ≥ 0. They are thus both of type ϕ, as in (2), and have expansions as in (1) with $k_\infty = 0$, the k_r all having the sign of A, as also has c for that one of the two functions which has degree 0. The form (4) for f and $1/f$ follows at once.

Conversely, if $g(\lambda)$ is of type (4), with c and k_r of the same sign, then

$$-\lambda g = cx + c' + \sum_{r=1}^{\mu} k_r\omega_r^2/(\omega_r^2 - x) \quad (c' = -\Sigma k_r)$$

* This is of order n, with its (r, s) element equal to $x + b_r (r = s)$, $a_s (r + 1 = s)$, $c_r (r = s + 1)$, otherwise 0, where $a_r > 0$, $c_r > 0$.

which is of type (1), whence the present converse part follows from that of theorem I.

To apply theorem II to Δ_n of p. 94,* we have, using λ instead of x,

$$\Delta_{n+1}=\lambda\Delta_n+a_{n+1}c_n\,.\,\Delta_{n-1},$$

or

$$\Delta_{n+1}/\Delta_n=\lambda+a_{n+1}c_n\,.\,\Delta_{n-1}/\Delta_n.$$

Then if Δ_{n-1}/Δ_n is of type (4), with positive coefficients, and strictly alternating poles and zeros on the imaginary axis, so is Δ_{n+1}/Δ_n. Since $\Delta_2/\Delta_1=\lambda+a_2c_1/\lambda$, which is of type (4) with $\omega_1=0$, the result for Δ_{n+1}/Δ_n, and likewise for Δ_n/Δ_{n+1}, follows by induction.

The results on p. 95 of the paper, that the roots of F_r and G_{r-1}, and of F_r and G_r, alternate on the imaginary axis, amount to saying that $\Delta_{2r}/\Delta_{2r-1}$ and $\Delta_{2r}/\Delta_{2r+1}$ have alternating imaginary poles and zeros, which has just been proved using theorem II. But we can in fact go further than this. For

$$\Delta_{n+2}=(\lambda^2+a_{n+2}c_{n+1})\Delta_n+a_{n+1}c_n\,.\,\lambda\Delta_{n-1},$$

that is,

$$\Delta_{n+2}/\lambda\Delta_n=\lambda+a_{n+2}c_{n+1}/\lambda+a_{n+1}c_n\,.\,\Delta_{n-1}/\Delta_n$$

which is seen also to be of type (4), so that the roots of $\lambda\Delta_n$ and Δ_{n+2} alternate, that is, those of F_r and F_{r+1} alternate, and also those of G_r and G_{r+1}.

It may be added in conclusion that functions of type (4) are very important in electrical network theory, where they are termed " reactance functions ", since they represent the impedance or admittance of any two-terminal reactance network, that is, one composed solely of coils and condensers, without any resistance elements. Theorem II expresses results which are well-known for such networks. Similarly, theorem I, if slightly modified, expresses known results for networks composed of coils and resistances only, or of condensers and resistances only. A. TALBOT.

* This is of order n with (r, s) element equal to $x\,(r=s)$, $a_s\,(r+1=s)$, $-c_r\,(r=s+1)$, otherwise 0, where $a_s>0$, $c_r>0$.

1714. When you toss a coin to decide who is going to pay the bill, let your companion do the calling. " Heads " is called seven times out of ten. The simple law of averages gives the man who listens a tremendous advantage.—Quoted in *Reader's Digest*, September 1950, from *The Saturday Review of Literature*. [Per Mr. F. Horner.]

1715. I remembered too that night which stands in the centre of the Thousand and One Nights, when Queen Schcherazade (by some magical aberration of the copyist) begins to relate word for word the story of the Thousand and One Nights, incurring the danger of arriving once more at the night in which she tells the story of the Nights, and thus continuing to infinity. —J. L. Borges, " The Garden of Forking Paths," in *The Queen's Awards*, Third Series (1950), p. 307. [Per Prof. E. H. Neville.]

1716. John Playfair happened to be at Foston at the same time as Mrs. Apreece, whose vivacity had made her the toast of Edinburgh. . . . The contrast between them was too much for Sydney [Smith], who instantly pretended that she had succumbed to the charms of the mathematical professor and wrote to Lady Holland : " It was wrong at her time of life to be circumvented by Playfair's diagrams ; but there is some excuse in the novelty of the attack, as I believe she is the first woman that ever fell a victim to algebra, or that was geometrically led from the paths of virtue."—Hesketh Pearson, *The Smith of Smiths*, Ch. VIII.

STEREOMETRY.

BY VINCENZO G. CAVALLARO.

IN this note I give the important trigonometric relations between the Brocard and Steiner angles of the cross-section of a triangular prism and the angle which the plane of this cross-section makes with the plane cutting the prism in an equilateral triangle. To facilitate a comprehension of the whole problem, I also give a complete and brief solution of the known problem for the equilateral section of the prism, deducing the equations of the equilateral case from the equations for the general case in which the section is a triangle similar to a given triangle.

1. In a triangle of sides $BC=a$, $CA=b$, $AB=c$, let ω be the Brocard angle, ω_1 and ω_2 the first and second Steiner angles, τ and τ' the Torricelli segments,* Δ the area. Then †

$$\cot \omega_1 = \cot \omega + \sqrt{(\cot^2\omega - 3)},$$

$$\cot \omega_2 = \cot \omega - \sqrt{(\cot^2\omega - 3)}. \quad \text{..........................(A)}$$

We note that the maximum of ω is $30°$, $\tan \omega \leqslant 1/\sqrt{3}$, $\omega_1 < \omega < \omega_2$. From the formulae (A) it follows that

$$\tan \omega = 2 \tan \omega_1/(1+3\tan^2 \omega_1) = 2\tan \omega_2/(1+3\tan^2\omega_2). \quad \text{...............(B)}$$

$$\cot(\omega_1+\omega_2) = \tan \omega, \quad \omega+\omega_1+\omega_2 = 90°, \quad \tan \omega_1 \tan \omega_2 = \tfrac{1}{3}.$$

2. From the geometry of the triangle it is known that

$$2\tau^2 = (a^2+b^2+c^2) + 4\Delta\sqrt{3},$$

$$2\tau'^2 = (a^2+b^2+c^2) - 4\Delta\sqrt{3},$$

$$16\Delta^2 = 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4).$$

We deduce that

$$a^2+b^2+c^2 = \tau^2+\tau'^2, \quad 4\Delta\sqrt{3} = \tau^2 - \tau'^2, \quad \text{..............(1)}$$

$$(a^4+b^4+c^4) - (a^2b^2+b^2c^2+c^2a^2) = \tau^2\tau'^2, \quad \text{..................(2)}$$

$$(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 = 2\tau^2\tau'^2, \quad \text{..................(3)}$$

$$(a^4+b^4+c^4) = \tfrac{1}{3}(\tau^4+4\tau^2\tau'^2+\tau'^4),$$

$$(a^2b^2+b^2c^2+c^2a^2) = \tfrac{1}{3}(\tau^4+\tau^2\tau'^2+\tau'^4). \quad \text{...(4)}$$

3. We also have ‡

$$\tan\omega = \frac{1}{\sqrt{3}}\,\frac{\tau^2-\tau'^2}{\tau^2+\tau'^2}, \quad \tan\omega_1 = \frac{1}{\sqrt{3}}\,\frac{\tau-\tau'}{\tau+\tau'}, \quad \tan\omega_2 = \frac{1}{\sqrt{3}}\,\frac{\tau+\tau'}{\tau-\tau'}. \quad \text{......(5)}$$

4. *Section of prism.* Now let ABC be the cross-section of a triangular prism, and let FGH be a given triangle of sides $GH=f$, $FH=g$, $FG=h$. A plane passing through A cuts the prism in the triangle APQ. Suppose that this triangle is similar to the triangle FGH. (The reader may easily supply a figure: we

* See V. G. Cavallaro, "On Lemoine's Ellipse," *Math. Gazette*, No. 310, p. 266 (December, 1950). For the Steiner angles, reference may be made to the classic *Traité de Géométrie* of Rouché and Comberousse, Vol. I, 8th edition, 1935, p. 477 (Gauthier-Villars, Paris), Note III, *Géométrie du triangle*, by J. Neuberg.

† See Rouché et Comberousse, *loc. cit.*, p. 477.

‡ V. G. Cavallaro, *Mathesis*, 1938, No. 9–10. It is known (see the article by Cavallaro cited in the first footnote) that the non-focal axis $2y$ of Lemoine's ellipse is given by the formula $2y = 2\sqrt{(\tfrac{1}{3}\Delta \tan\omega)}$. Now, by the equations (1) and (5), we have, on substitution,

$$2y = (\tau^2-\tau'^2)/3\sqrt{(\tau^2+\tau'^2)}.$$

Hence we have $2y$ in terms of the Torricelli segments.

suppose that P and Q are on those edges of the prism through B and C respectively, and that $CQ > BP$). Writing $AP = z$, $BP = x$, $CQ = y$, then since APQ and FGH are similar,

$$PQ/f = AQ/g = AP/h = z/h$$

or

$$AP = z, \quad PQ = zf/h, \quad AQ = zg/h. \qquad \text{(i)}$$

Then by Pythagoras' theorem

$$c^2 + x^2 = z^2, \quad b^2 + y^2 = z^2g^2/h^2, \quad a^2 + (y - x)^2 = z^2f^2/h^2. \qquad \text{(j)}$$

These equations give the value of z and hence, by (i), the sides of the triangle APQ, and with the values of x and y we can determine the points P and Q such that the triangle APQ is similar to FGH.

Equilateral section. If, in particular, the given triangle FGH is equilateral, so that $f = g = h$, the equations (j) reduce to

$$z^2 = c^2 + x^2, \quad z^2 = b^2 + y^2, \quad z^2 = a^2 + (y - x)^2$$

whence

$$c^2 = a^2 + y^2 - 2xy, \quad b^2 = a^2 + x^2 - 2xy. \qquad (\text{j}_1)$$

Now in this case the triangle APQ is to be equilateral, of side ξ, say, and we have

$$c^2 + x^2 = b^2 + y^2 = z^2 = \xi^2. \qquad (\text{j}_2)$$

Replacing y in the first equation (j) by its value from the second, we have

$$3x^4 - 2(a^2 + b^2 - 2c^2)x^2 - (a^2 - b^2)^2 = 0$$

whence

$$3x^2 = (a^2 + b^2 - 2c^2) \pm 2\surd\{(a^4 + b^4 + c^4) - (a^2b^2 + b^2c^2 + c^2a^2)\}$$

and similarly

$$3y^2 = (a^2 + c^2 - 2b^2) \pm 2\surd\{(a^4 + b^4 + c^4) - (a^2b^2 + b^2c^2 + c^2a^2)\}$$

Now it is easily seen that the quantity

$$4(a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2)$$

is greater than either $(a^2 + b^2 - 2c^2)^2$ or $(a^2 + c^2 - 2b^2)^2$. Hence in order that x^2 and y^2 should be positive, it is necessary to take the positive sign in the ambiguities preceding the root sign. Consequently, by the relations (1) and (2) of § 2,

$$x^2 = \tfrac{1}{3}(\tau^2 + \tau'^2 - 3c^2 + 2\tau\tau') = \tfrac{1}{3}(\tau + \tau')^2 - c^2,$$

$$y^2 = \tfrac{1}{3}(\tau^2 + \tau'^2 - 3b^2 + 2\tau\tau') = \tfrac{1}{3}(\tau + \tau')^2 - b^2,$$

whence

$$x^2 + c^2 = y^2 + b^2 = \tfrac{1}{3}(\tau + \tau')^2,$$

and so, by the relations (j_2),

$$\xi = \tfrac{1}{3}(\tau + \tau')\surd 3. \qquad (\alpha)$$

Thus: *the side ξ of the equilateral triangle APQ, the section of the prism by the plane APQ, is equal to one-third the side of the equilateral triangle inscribed in the circle whose radius is the sum of the Torricelli segments of the triangle which is the cross-section of the prism.*

5. *The angle between the planes ABC, APQ.* If ϕ is the angle between the planes ABC and APQ, we have

$$(\text{area } ABC) = (\text{area } APQ)\cos\phi,$$

whence, using equations (α) and (1),

$$\cos\phi = 4\Delta/\xi^2\surd 3 = (\tau - \tau')/(\tau + \tau'). \qquad (\beta)$$

6. *A noteworthy relation between the angles ω, ω_1, ω_2 and the angle ϕ.*

6·1. The formulae (β) of § 5 and (5) of § 3 give

$$\cos\phi = \sqrt{3}\,.\tan\omega_1\,;\quad \sec\phi=\sqrt{3}\,.\tan\omega_2. \qquad\qquad (\gamma)$$

Note. The formulae (α) and (5) give

$$\tan\omega_1=(\tau-\tau')/3\xi,\ \tan\omega_2=\xi/(\tau-\tau'),$$
$$\xi=\tfrac{1}{3}(\tau-\tau')\cot\omega_1=(\tau-\tau')\tan\omega_2.$$

6·2. From the formulae (γ) of § 6 and (B) of §1 we have

$$2\cot\omega=\sqrt{3}.\,(\sec\phi+\cos\phi).$$

V. G. C.

1717. I shall now work hard at my Mathematics in which my Coach gives me hopes of getting a low Wrangler's degree ; I am now reading the sesame of Mathematics, the Differential Calculus.—Letter from Sir William Harcourt, quoted in A. G. Gardiner, *Life of Sir William Harcourt*, I, p. 54 (1923).

1718. *Loss of time involved in walking up hills when cycling.*

. . . the coasting never really compensates for the foot-slogging. One has only to consider the matter mathematically to see how this comes about. Assuming that my normal speed on the level is 10 m.p.h. this is reduced to 3 m.p.h. when I walk up hill, or perhaps to a mere 2 m.p.h. if the hill is a long one and I have to stop for a rest half way up it. My speed in fact goes down to perhaps one-fifth of normal. When coasting, on the other hand, I certainly do not increase my normal speed five times! I may double it but no more.—Earl of Cardigan, *I Walked Alone* (London, 1950). [Per Mr. G. A. Bull.]

1719. "That Midnight Kiss," a Pasternak musical film of the kind that Hollywood turns out by the mile and then cuts up into suitable 9,000-foot lengths.—New Zealand Broadcasting Service, *The Listener*, May 26, 1950. [Per Mr. M. A. Bull.]

1720. Of the siege machines working by torsion the best type was the mangon, which . . . cast the rock or ball with a high elliptic trajectory.—Sir Charles Oman, *The Art of War in the Middle Ages* (2nd edition, 1924), I, pp. 136–7.

1721. Much more delightful were the elaborate calculations of a candidate who, being set to paper and carpet a room of ample dimensions, of which the exact measurements were given, sent me in a bill for some thousand pounds. The process by which this was reached was logical. The room is a solid void. If carpet costs 3s. 3d. per square yard, and has to be used in a cubic space, then cubic carpet must cost 9s. 9d. per yard. So he filled the room from floor to roof—the height being given—with solid blocks of cubic carpet. He then, knowing that wall paper was to cost 1s. 6d. per piece, made out that cubic wallpaper must run to 4s. 6d. per block, and filled the room a second time with these rectangular masses.—Sir Charles Oman, *Memories of Victorian Oxford*, p. 213.

1722. It is probably scientific licence, however, which causes Sir Ian Heilbron to suggest that our children will become Methuselahs (969 years), though the purists say they used lunar years in those days—making him a young upstart of 243.—*News Chronicle*, March 9, 1950. [Per Mr. R. E. Gundry.]

MATHEMATICAL NOTES.

2297. *Approximate construction of the regular heptagon.*

Note 1617 (Vol. 26, No. 271) by Mr. H. Martyn Cundy gives a general and quite practical method of inscribing a regular polygon in a given circle, and I have often found that device an interesting exercise in geometrical drawing. (Since all drawing is subject to error, is it a sin to admit the error in advance?).

In presenting this sort of material to a junior form recently, one of the pupils hit upon the following particularly simple heptagon-construction, derived from the usual construction for the hexagon.

Join any pair of alternate vertices of the hexagon, and let this chord be of length $2l$. Then $l=\frac{1}{2}r\sqrt{3}$, where r is the radius of the circle.

i.e., $$l=0{\cdot}866r.$$

Now, by calculation from the centre-angle of the heptagon, the length of the side of the heptagon $=0{\cdot}8678r$.

Hence, the length l may be taken as the side of the inscribed heptagon, with an error of only 0·2% approximately. B. J. Banner.

2298. *A digital puzzle.*

The following intriguing puzzle of unknown origin was propounded in the *New Statesman*, p. 761, Dec. 24th, 1949, by J. Bronowski. There it was suggested that the solution would require an all night sitting. However, this warning was probably intended for those unacquainted with the theory of congruences.

The puzzle consists of finding the smallest integral number such that if the digit on the extreme left is transferred to the extreme right, the new number so formed is one and a half times the original number. The answer is surprisingly large.

Let the original number be $10^n a_n+10^{n-1}a_{n-1}+\ldots+a_0$ where a_0, a_1, etc. are restricted to the values 0, 1, 2...9. Then, according to the conditions of the problem,

$$3(10^n a_n+10^{n-1}a_{n-1}+\ldots+a_0)=2(10^n a_{n-1}+10^{n-1}a_{n-2}+\ldots+10a_0+a_n).$$

Thus

$$17(10^{n-1}a_{n-1}+10^{n-2}a_{n-2}+\ldots+a_0)=(3\,.\,10^n-2)a_n,$$

and so $$3\,.\,10^n\equiv 2(\text{mod } 17).$$

Now $$10\equiv -7,\quad 10^2\equiv -2,\quad 10^4\equiv 4,\quad 10^8\equiv -1.$$

Hence $$10^{15}\equiv(-1)(4)(-2)(-7)\equiv -5$$

and since $$3\equiv -14,$$

we have $$3\,.\,10^{15}\equiv(-5)(-14)\equiv 2.$$

Hence $$10^{14}a_{14}+10^{13}a_{13}+\ldots+a_0=\tfrac{1}{17}(3\,.\,10^{15}-2)a_{15}$$
$$=176470588235294a_{15}.$$

The left-hand side is necessarily a 15-figure number. This can only happen if $a_{15}<6$.

The smallest integral number is obtained by putting $a_{15}=1$. Note that no value of $n<15$ will satisfy the congruence. Hence the required number is 1,176,470,588,235,294 and the transformed number is 1,764,705,882,352,941, agreeing with the answer given in the *New Statesman*. J. H. Clarke.

2299. *On Notes* 2023 *and* 2094.

Dr. Busbridge's extensions of the result quoted by Mr. Parameswaran reminded me of the following, obtained some years ago. It is easy to show that

$$\frac{1}{s}\binom{n}{r+s-1}-\frac{1}{s+1}\binom{n}{r+s}\binom{s+1}{s}+\frac{1}{s+2}\binom{n}{r+s+1}\binom{s+2}{s}-\dots$$
$$+\frac{(-)^{n-r-s+1}}{n-r+1}\binom{n-r+1}{s}=\frac{1}{s}\binom{n-s}{r-1};\quad\dots\dots\dots\dots(1)$$

whence

$$\binom{n}{r}\{(1+x)-1\}-\tfrac{1}{2}\binom{n}{r+1}\{(1+x)^2-1\}+\dots+\frac{(-)^{n-r}}{n-r+1}\{(1+x)^{n-r+1}-1\}$$
$$=\binom{n-1}{r-1}x-\tfrac{1}{2}\binom{n-2}{r-1}x^2+\tfrac{1}{3}\binom{n-3}{r-1}x^3-\dots+\frac{(-)^{n-r}}{n-r+1}x^{n-r+1},\quad\dots(2)$$

and putting $x=-1$, we have (with an induction)

$$\binom{n}{r}-\tfrac{1}{2}\binom{n}{r+1}+\dots+\frac{(-)^{n-r}}{n-r+1}=\binom{n-1}{r-1}+\tfrac{1}{2}\binom{n-2}{r-1}+\dots+\frac{1}{n-r+1}$$
$$=\binom{n}{r-1}\left\{\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\dots+\frac{1}{n}\right\}.\quad\dots\dots\dots\dots(3)$$

Thus

$$\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\dots+\frac{1}{n}=(r-1)!\sum_{s=1}^{n-r+1}\frac{(-)^{s-1}\binom{n-r+1}{s}}{s(s+1)\dots(s+r-1)},\quad\dots\dots(4)$$

which is a variant of Dr. Busbridge's generalization.

The series in (4) may be exhibited in the form

$$\frac{(r-1)!}{n!}\times\begin{vmatrix} 2r+1 & -(r+1) & 0 & . & . & . & . & 0 \\ -(r+1) & 2r+3 & -(r+2) & . & . & . & . & 0 \\ 0 & -(r+2) & (2r+5) & . & . & & . & 0 \\ 0 & 0 & -(r+3) & . & . & & . & 0 \\ . & . & . & . & . & & . & . \\ . & . & . & . & . & & . & -(n-1) \\ 0 & 0 & 0 & . & & & -(n-1) & (2n-1) \end{vmatrix}$$

Many special identities can be obtained from (2) or its integrals, by giving particular values to x and r.

From the general result (3) it is not difficult to deduce the following curious relation :

$$2^{n-1}\left\{1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{(-)^{n-1}}{n}\right\}$$
$$=\left(1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{k}\right)+\binom{n+1}{2}\left(\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{k}\right)$$
$$+\binom{n+1}{4}\left(\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{k}+\dots+\right)+\dots+\binom{n+1}{k-1}\frac{1}{k},$$

where k is the greatest odd integer not exceeding n. B. A. SWINDEN.

2300. *A note on Conics having three points common.*

We define a set of three points A' as the residual intersection of two conics

through the fixed points B, A and C, A respectively, and further define the sets B' and C' as the residual intersections of these conics with one drawn through B and C. The three conics defined by the sets of five points A', B, C ; B', C, A ; C', A, B do not have three points in common in general. In this note we show that the necessary and sufficient condition that they should have three points D, E, F in common is that the conics A', B, C and B', B, C', C shall have a degenerate ϕ envelope.* Given the two conics A', A, B', B and A', A, C', C, the remaining conic B', B, C', C may be chosen to satisfy this condition.

It is very easy to show† that if any straight line is drawn through A to cut the conics A', A, B', B and A', A, C', C in X and Y respectively, then the lines XB and YC will meet in a point a on the conic A', B, C. Produce the lines XBa, YCa to cut the conic B', B, C', C again in the points Z_1 and Z_2 respectively, and join Z_1C, Z_2B to intersect in a', and to cut XY in the points b and c respectively. Defining the points D, E, F as the remaining intersections of the conics B', C, A and C', A, B ; the theorem above shows that b and c lie on these two conics respectively, and hence that the locus of a' is the conic defined by D, E, F, B, C. The required necessary and sufficient condition is therefore that a' shall lie also on the conic A', B, C. The locus of a' is the same as that of a if and only if the conics A', B, C and B', B, C', C have a degenerate ϕ envelope. It follows that the conics B', C, A and C', C, A', A have a degenerate ϕ envelope, and also the conics C', A, B and A', A, B', B.

Analytically we may take the triangle ABC as triangle of reference, and the three conics as

$$f_1yz + g_1zx + h_1xy + a_1x^2 = 0,$$
$$f_2yz + g_2zx + h_2xy + b_2y^2 = 0,$$
$$f_3yz + g_3zx + h_3xy + c_3z^2 = 0.$$

Taking XAY as the line $y=0$, the degenerate cubic formed by the conic A, B, C and the line AC is then

$$(f_2yz + g_2zx + h_2xy + b_2y^2)(g_3x + c_3z) = (f_3yz + g_3zx + h_3xy + c_3z^2)(f_2y + g_2x),$$

and the equation of the conic A', B, C is

$$(h_2x + b_2y)(g_3x + c_3z) - (f_3z + h_3x)(f_2y + g_2x) = 0,$$

which we may write as

$$F_1yz + G_1zx + H_1xy + A_1x^2 = 0,$$

and similarly for the conics B', C, A and C', A, B.

Defining the points D, E, F as before, the equation of the conic D, E, F, B, C may be written as

$$(F_2F_3 - B_2C_3)yz + (G_2F_3 - H_2C_3)zx + (F_2H_3 - B_2G_3)xy + (G_2H_3 - H_2G_3)x^2 = 0,$$

which is the conic A', B, C if the ratio of the coefficients in the equations are equal. These give the single condition

$$h_1G_1 + g_1H_1 = a_1F_1 + f_1A_1,$$

or one of its equivalents

$$f_2H_2 + h_2F_2 = b_2G_2 + g_2B_2,$$
$$g_3F_3 + f_3G_3 = c_3H_3 + h_3C_3.$$

L. E. PRIOR.

* H. F. Baker, *Principles of Geometry*, Vol. II, pp. 42–44 ; or H. G. Green, *Mathematical Gazette*, Vol. XXVIII, No. 278, p. 3, where the case of degeneracy is considered in a much simpler manner.

† By (1, 1) correspondence, or by projecting the conics concerned into circles.

2301. *Simple dynamics and the value of g.*

Let us assume that the Earth is a uniform sphere without orbital motion or rotation : let γ = the attraction of the Earth for a unit of mass on its surface γ is the same all over the Earth. The Earth, however, rotates once about its axis in 86164 secs. and to supply the necessary centripetal force on a particle at rest on the surface of the Earth some small part of the γ is used up and the remainder gives the resultant gravitation attraction which we call g. Another consequence of this is that, in general, a plumb line does not point to the centre of the Earth.

The relation between γ and the value of g at any latitude λ, say g_λ, is proved in many textbooks (*e.g.* Wagstaff's *Properties of Matter*). It is

$$g_\lambda = \gamma - \omega^2 R \cos^2 \lambda,$$

where R = the radius of the Earth and ω = the angular velocity of rotation. In C.G.S. units if we assume a value of g at the Poles of 982·00 cms. sec^{-2} (and this is also the value of γ), then

$$g_\lambda = (982{\cdot}00 - 3{\cdot}39 \cos^2 \lambda) \text{ cm. sec}^{-2}.$$

The centre of the Earth being assumed at rest (*i.e.* neglecting any orbital motion) the kinetic energy of a particle of mass m on the surface of the Earth in latitude $\lambda = \frac{1}{2} m\omega^2 (R \cos \lambda)^2$. If this particle is lifted up a height h (h being small in comparison with R), and put on a bracket fixed to the Earth its kinetic energy becomes $\quad \frac{1}{2} m\omega^2 (R+h)^2 \cos^2 \lambda$.

Thus, its increase of kinetic energy $= \frac{1}{2} m\omega^2 (R+h)^2 - R^2 \cos^2 \lambda$

$$= m\omega^2 R \cos^2 \lambda \,.\, h \quad \text{very nearly}$$

$$= m(\gamma - g_\lambda)h.$$

Similarly, if a particle of mass m originally on a bracket at height h falls into a pocket on the surface of the Earth its kinetic energy is decreased by $m(\gamma - g_\lambda)h$. Its potential energy decreases by $mg_\lambda h$. Therefore, the total loss of mechanical energy (which may be turned into heat by impact in the pocket) $= m(\gamma - g_\lambda)h + mg_\lambda h = m\gamma h$, and this is the same all over the surface of the spherical Earth.

If this reasoning is correct, the result is interesting and possibly new.

At the equator, $g = 978{\cdot}61$ cm. sec^{-2} and at every place $\gamma = 982{\cdot}00$ cm. sec^{-2} (assumed value, see above), therefore the difference between the equatorial value of g and the value of γ is 3·39 cm. sec^{-2}. This difference is of the order of 0·3 per cent. So that if the heating effect produced by the impact of a falling body at the equator is calculated from the equation, energy available $= mgh$, it is in error by this amount.

In the above, we have assumed a spherical Earth. Modifications may be made to suit the actual ellipsoidal Earth. JOHN SATTERLY.

2302. *The constant-coefficient linear equation : equal roots.*

Engineering students need to solve the usual second-order linear differential equation with constant coefficients for the case of equal roots. They may not need or proceed far enough to appreciate the usual operational formulae. The following method appears to meet their requirements.

We need to solve

$$\frac{d^2x}{dt^2} + 2a \frac{dx}{dt} + a^2 x = 0.$$

This is

$$\frac{d^2x}{dt^2} + a \frac{dx}{dt} = -a \left(\frac{dx}{dt} + ax \right),$$

or

$$\left(\frac{d^2x}{dt^2}+a\frac{dx}{dt}\right)\Big/\left(\frac{dx}{dt}+ax\right)= -a.$$

Integrating,

$$\log\left(\frac{dx}{dt}+ax\right)= -at+\log A,$$

whence

$$\frac{dx}{dt}+ax=Ae^{-at};$$

this may be written

$$e^{at}\left(\frac{dx}{dt}+ax\right)=A,$$

or

$$\frac{d}{dt}(xe^{at})=A.$$

Thus

$$xe^{at}=At+B,$$

and

$$x=(At+B)e^{-at}.$$

J. PEDOE.

2303. *The ejection of a cork from a bottle.*

Although this phenomenon is so well known, no simple discussion of it seems as yet to have been given. When certain simplifying assumptions of a physical character are made, the problem becomes a definitely mathematical one. It therefore seems worthwhile to include the present note.

Let F_0 be the greatest force which will just fail to remove the cork in a horizontal position. If the pressure of the neck of the bottle upon the cork is uniform, the force of friction on the cork when it has been pushed out a distance x is $F_0(a-x)/a$, where a is the distance through which it has to travel before leaving the bottle.

Motion will not commence until the pressure inside the bottle exceeds $F_0/A + W$, where A is the area of cross-section of the cork, assumed cylindrical and W is its weight. We measure the time t from the instant when the pressure within the bottle reaches this critical value ; then, initially $x=0$, $\dot{x}=0$, and $\ddot{x}=0$.

After time t, the pressure of gas urging the cork forward is due to :

(1) the expansion of the mass of gas present at time $t=0$ due to the motion of the cork : let this be p_1, say ;

(2) the additional pressure p_2 (say), due to the gas generated by the enclosed fermenting liquid during this time t. Let this be p_2.

We shall now assume that Dalton's Law of Partial Pressures may be applied here,* so that the total pressure on the cork face is p_1+p_2.

Evaluation of the pressure

If (as will usually be the case) the ejection of the cork takes place quickly, the expansion of the gas present at $t=0$ will be effectively adiabatic. Hence, if P, V be the initial pressure and volume respectively,

$$p_1=P\,.\,V^{\gamma}/(V+Ax)^{\gamma}. \qquad\qquad (1)$$

Let R be the rate† per cent. per second at which the pressure is increasing

* According to physical advice which has been obtained, this is legitimate.

† It can be shewn from the gas laws that R is also the percentage rate of increase of the mass of the gas.

within the flask at time $t=0$; then

$$R=100(dp/p\,dt)_0. \qquad \text{.....(2)}$$

Now

$$(dp/dt)_0=(dp_1/dt)_0+(dp_2/dt)_0$$

$$=(dp_2/dt)_0, \qquad \text{.....(3)}$$

since $\dot{x}=0$ when $t=0$.

Hence, $$R=100(dp_2/dt)_0\,.\,P^{-1} \qquad \text{.....(4)}$$

If the time of ejection (T say) is sufficiently small we can regard dp_2/dt as nearly constant during the ejection; hence, approximately

$$p_2=(PR/100)t \quad (0<t<T) \qquad \text{.....(5)}$$

Hence, by the Law of Partial Pressures, the equation of motion is

$$\dot{v}=(g/W)\,.\,\{AP\,.\,V^\gamma/(V+Ax)^\gamma+A(PR/100)t-F_0(1-x/a)-W\} \qquad \text{.....(6)}$$

but

$$PA=F_0+W, \qquad \text{.....(7)}$$

hence this equation becomes

$$v=\{g(F_0+W)/W\}\{(1+Ax/V)^{-\gamma}+Rt/100-1+F_0x/a(F_0+W)\}. \qquad \text{.....(8)}$$

In this article, we shall consider only the case when $aA \ll V$. This is the case when the bottle is not too full. The equation of motion then approximates to

$$\dot{v}=\{g(F_0+W)/W\}\{Rt/100+(F_0x)/a(F_0+W)\}. \qquad \text{.....(9)}$$

Differentiating, therefore,

$$\ddot{v}=(gF_0/Wa)\{v+a(F_0+W)R/100F_0\}. \qquad \text{.....(10)}$$

Since v and $\dot{v}$ both vanish at $t=0$, the solution of this equation is

$$v=b(\cosh\mu t-1), \qquad \text{.....(11)}$$

where

$$\mu=\surd(gF_0/aW), \qquad b \doteq (aR/100)\{1+W/F_0\} \qquad \text{.....(11a)}$$

Integrating, we have

$$x=b\left\{\frac{1}{\mu}\sinh\mu t-t\right\} \qquad \text{.....(12)}$$

Hence, T is given by

$$\frac{1}{\mu}\sinh\mu T-T=(100/R)\{1+W/F_0\}^{-1}, \qquad \text{.....(13)}$$

provided that T is small enough for the approximation (5) to be valid.

Now, if μT is fairly large ($>$abt 3, say) then (13) gives, approximately,

$$T=(\mu^{-1})\log(200\mu\sigma/R), \qquad \text{.....(14)}$$

where

$$\sigma=(1+W/F_0)^{-1}. \qquad \text{.....(14a)}$$

$$\doteqdot 1 \text{ in most practical cases.}$$

Therefore, if $W/F_0 \ll 1$, which will usually be the case, we have the approximate formula

$$T=\mu^{-1}\log(200\mu/R) \qquad \text{.....(15)}$$

provided $\mu T>$abt 3, *i.e.* provided that

$$R<10\mu \quad \text{(about).} \qquad \text{.....(16)}$$

This condition is likely to be amply satisfied as R is not likely to exceed (say) 10% per second, in the case of most fermenting liquors, and μ is cer-

tainly greater than order 16 $\sec^{-1}$ as a is not likely to exceed about an inch, F_0 certainly exceeds W.

Thus, for $F_0 \gg W$, the time of ejection is approximately as given by (14) and, since R is almost certain to be very large compared with $200\mu e^{-\mu}$, μ being greater than about 16 $\sec^{-1}$, T is likely to be very small in comparison with one second.

The velocity of ejection is then from (11), approximately $a\mu$, *i.e.*

$$\sqrt{(agF_0/W)}.$$

Typical numerical case

If $a=\frac{3}{4}''$, $F_0/W=20$, $R=4\%$ per sec., $\mu=101{\cdot}2\ \sec^{-1}$,
Vel. of ejection $=6{\cdot}32$ ft./sec.
Time of ejection $=0{\cdot}085$ seconds.

Note :

A better approximation to equation (9) is

$$v=\{g(F_0+W)/W\}\{Rt/100+[F_0/a(F_0+W-\gamma A/V]x\},$$

giving

$$\ddot{v}=gF_0/Wa\{v[1-\gamma Aa(F_0+W)/F_0V]+a(F_0+W)R/100F_0\}.$$

If $W \ll F_0$, this is approximately

$$\ddot{v}=(gF_0/Wa)(1-\gamma Aa/V)\{v+(aR/100)(1-\gamma Aa/V)^{-1}\}$$

and the solution is then as above, with

$$\mu=\sqrt{\{(gF_0/Wa)(1-\gamma Aa/V)\}}, \qquad b=(aR/100)(1-\gamma Aa/V)^{-1},$$

and so, with the same approximations, the terminal velocity $a\mu$ is slightly less.

F. H. N.

2304. *Newton's approximation.*

Suppose that $f(z)$ is analytic in some circular domain D about a simple zero z_0 and that $f'(z)$ does not vanish therein. Let a be any point of this domain. Then we have to consider whether the sequence defined by

$$a_n=a_{n-1}-f(a_{n-1})/f'(a_{n-1}), \quad a_1=a-f(a)/f'(a)$$

tends to a limit as n tends to infinity, and whether this limit is z_0. Now, by hypothesis,

$$\begin{aligned}0=f(z_0)&=f(a+z_0-a)\\&=f(a)+(z_0-a)f'(a)+\tfrac{1}{2}(z_0-a)^2f''(a)+\ldots.\end{aligned}$$

This Taylor series converges absolutely and uniformly since a is in D, and we have

$$a-z_0-f(a)/f'(a)=\{\tfrac{1}{2}(z_0-a)^2f''(a)+\ldots\}/f'(a),$$

that is,

$$a_1-z_0=(z_0-a)^2\{\tfrac{1}{2}f''(a)+\ldots\}/f'(a).$$

Similarly,

$$a_2-z_0=(z_0-a_1)^2\{\tfrac{1}{2}f''(a_1)+\ldots\}/f'(a_1),$$

and, generally,

$$a_r-z_0=(z_0-a_{r-1})^2\{\tfrac{1}{2}f''(a_{r-1})+\ldots\}/f'(a_{r-1}),$$

provided that $a_1, a_2, \ldots, a_r$ are all in D.

Now

$$|(a_1-z_0)/(a-z_0)|=|a-z_0|\,|\tfrac{1}{2}f''(a)+\ldots|/|f'(a)|$$

and since z_0 is a fixed number, $f'(a)\neq 0$, and $f(z)$ is analytic in D we may choose a so that $|a_1-z_0|<|a-z_0|$. Hence, a_1 is also in D. Now, let M be the maxi-

mum modulus of

$$\{\tfrac{1}{2}f''(z)+\ldots\}/f'(z)$$

on the circle $\quad |z-z_0|=|a-z_0|,$

then, taking $\quad |a-z_0|<1/M,$

it follows that

$$|(a_2-z_0)/(a_1-z_0)|<|a-z_0|\,|\,\tfrac{1}{2}f''(a_1)+\ldots|/|f'(a_1)|$$
$$<|a-z_0|\,M<1$$

by the maximum modulus theorem, since a_1 is inside the circle

$$|z-z_0|=|a-z_0|.$$

Hence $\quad |a_2-z_0|<|a_1-z_0|,$

so that a_2 is also in D. Let

$$|a_1-z_0|=k\,|a-z_0|,$$

so that $k<1$. Then, by successive applications of the maximum modulus theorem

$$|a_2-z_0|<k\,.\,M\,|a-z_0|^2,$$
$$|a_3-z_0|<k^2\,.\,M\,|a-z_0|^2,$$

and generally

$$|a_{r+1}-z_0|<k^r\,.\,M\,|a-z_0|<k^r\,|a-z_0|,$$

since $\quad |a-z_0|<1/M.$

Hence $a_r\to z_0$ as $r\to\infty$.

For the case of the real variable, suppose $f(0)=0$, $f'(x)\neq 0$ in $|x|\leqslant a$, and that $f''(x)$ exists in $|x|<a$. Then, by the second mean value theorem

$$a_1=a-f(a)/f'(a)$$
$$=-\tfrac{1}{2}a^2f''(\theta a)/f'(a).\quad (0<\theta<1)$$

Hence we can take a^2 so small that $|a_1/a|<1$. In fact, if we take $a<\sqrt{(2m/M)}$, where m is the least value of $|f'(x)|$ and M is the greatest value of $|f''(x)|$ in $|x|<a$, we have

$$\overline{\lim}\,|a_{r+1}/a_r|<1$$

and so $a_r\to 0$ as $r\to\infty$. F. H. NORTHOVER.

2305. *The cube of a polynomial as a determinant of order* 3.

The polynomial in the indeterminate x of degree n

$$P(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n,\quad (a_0\neq 0)$$

where the coefficients a_j are positive or negative integers or zero, having no common factor, can be written

$$x^m(a_0x^h+a_1x^{h-1}+\ldots+a_{n-m})+(a_{n-m-1}x^{m-1}+\ldots+a_n)$$

where m is any positive integer less than n and $m+h=n$. Briefly

$$P(x)=x^mq+r. \qquad \text{(i)}$$

If we set $\quad U=x^m,\quad u=1,$

$$I=x^m-r,\quad i=q+1,$$
$$N=r,\quad n=-q,$$

then $Uq+ur=Iq+ir=P(x)$, identically, and $Nq+nr=0$.

It is easily seen that the absolute value of each of the three determinants which follow is $P(x)$.

$$\begin{vmatrix} U & u \\ N & n \end{vmatrix} = \begin{vmatrix} U & u \\ I & i \end{vmatrix} = \begin{vmatrix} I & i \\ N & n \end{vmatrix} = P(x).$$

We now give the algorithm : the absolute value of the determinant

$$\begin{vmatrix} U^2 & Uu & u^2 \\ N^2 & Nn & n^2 \\ I^2 & Ii & i^2 \end{vmatrix} = \{P(x)\}^3.$$

For, on expansion and simplification, the determinant

$$= (Un - Nu)(Ni - In)(Iu - Ui)$$
$$= P(x) \,.\, P(x) \,.\, P(x) = \{P(x)\}^3.$$

Numerical illustration

To express $(641)^3$ as a determinant of order 3 ;

(*a*) Let $U=100$, $u=1$, $I=59$, $i=7$, $N=41$, $n=-6$, $641=10^2.6+41$, then

$$\begin{vmatrix} 10000 & 100 & 1 \\ 1681 & -246 & 36 \\ 3481 & 413 & 49 \end{vmatrix} = (641)^3.$$

or (*b*) Let $U=10$, $u=1$, $I=9$, $i=65$, $N=1$, $n=-64$, $641=10.64+1$, then

$$\begin{vmatrix} 100 & 10 & 1 \\ 1 & -64 & 4496 \\ 81 & 585 & 4225 \end{vmatrix} = (641)^3.$$

M. Rumney.

2306. *Steggall's proof of an identity.*

In Note 2045 (XXXIII, pp. 37–8), Professor A. A. Krishnaswami Ayyangar gives a proof of the identity

$$4(ax+by+cz)^3 - 3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) - 54abcxyz$$
$$= 2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$$

which holds when $a+b+c=x+y+z=0$.

Strangely enough, on looking through an old mathematics scrap book of my schooldays, I found the following solution, which I copy exactly. There is a note that the solution is due to J. E. A. Steggall and the date given is 1915, but no reference is given to the source.

" Consider the equation of which the roots are $ax+by+cz$, $bx+cy+az$, $cx+ay+bz$: their sum is clearly 0, and that of their products two at a time is given by

$$S_2 = (x^2+y^2+z^2)(bc+ca+ab) + (yz+zx+xy)(a^2+b^2+c^2+bc+ca+ab)$$

which, by virtue of the given relations, is

$$-\tfrac{3}{4}(a^2+b^2+c^2)(x^2+y^2+z^2).$$

Their product is

$$(y^2z+z^2x+x^2y)(b^2c+c^2a+a^2b) + (yz^2+zx^2+xy^2)(bc^2+ca^2+ab^2)$$
$$+ (x^3+y^3+z^3)abc + 3abcxyz + (a^3+b^3+c^3)xyz.$$

Now $\quad b^2c+c^2a+a^2b+bc^2+ca^2+ab^2 \equiv A \equiv -3abc,$

$$b^2c+c^2a+a^2b-bc^2-ca^2-ab^2 \equiv B \equiv -(b-c)(c-a)(a-b)$$

and similarly for expressions in x, y, z. Hence, using also $x^3+y^3+z^3=3xyz$, the product is

$$\tfrac{1}{2}(A+B)\,.\,\tfrac{1}{2}(X+Y)+\tfrac{1}{2}(A-B)\,.\,\tfrac{1}{2}(X-Y)+9abcxyz$$
$$=\tfrac{1}{2}(AX+BY)+9abcxyz=\tfrac{27}{2}abcxyz+\tfrac{1}{2}BY.$$

Hence, $ax+by+cz$ is a root of the equation

$$t^3-\tfrac{3}{4}t(a^2+b^2+c^2)(x^2+y^2+z^2)$$
$$-\tfrac{1}{2}(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)-\tfrac{27}{2}abcxyz=0,$$

which on substitution and clearing of fractions is the identity required."

M. Rumney.

2307. *On sub-factorial n.*

With reference to Note 2183, of the Dec., 1950, issue, the following method of deriving the expression

$$n!\left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+\frac{(-1)^n}{n!}\right\}$$

for u_n, sub-factorial n, may be of interest.

Defining sub-factorial r, or u_r, as the number of ways in which, say, r numbered pegs can be placed in r similarly numbered holes, subject to the condition that none of the r pegs shall be in its corresponding hole, then ${}^nC_r u_r$ will be the number of ways in which n numbered pegs can be fitted into n similarly numbered holes subject to the condition that $(n-r)$ of the pegs shall be fitted each into its proper hole, while none of the remaining r pegs shall be so fitted. From this it follows that the sum

$$\sum_{r=0}^{r=n} {}^nC_r u_r$$

will represent the total number of unrestricted permutations of n things taken n at a time, *i.e.*

$$\sum_{r=0}^{r=n} {}^nC_r u_r = n!$$

By a formal analogy with the binomial expansion this may be written

$$(1+u)^n=n!$$

where u^r has replaced u_r.

Hence

$$u_n=u^n=\{(1+u)-1\}^n=(1+u)^n-n(1+u)^{n-1}+{}^nC_2(1+u)^{n-2}-\ldots$$
$$=n!-n(n-1)!+{}^nC_2(n-2)!-\ldots+(-1)^n$$
$$=n!\left\{\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}\right\}.$$

S. G. Horsley.

2308. *On Note* 2180 : *Spearman's rank-correlation coefficient.*

Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two rankings of n objects. Then Spearman's rank-correlation coefficient ρ has its sign changed if y_r is replaced by $(n+1-y_r)$ for $r=1, \ldots, n$. This is stated explicitly in Kendall's *Advanced Theory of Statistics* (1st edition, vol. I, § 16.11 (c)) and implied in Yule and Kendall's *Introduction to the Theory of Statistics* (12th edition, revised, formula 13.8). Hence, if $(Y_1, \ldots, Y_n)$ gives $\rho=-1$, then $(n+1-Y_1, \ldots, n+1-Y_n)$ gives $\rho=1$. But this requires that $n+1-Y_r=x_r$, *i.e.* that the two rankings

$$(x_1, \ldots, x_n) \quad \text{and} \quad (Y_1, \ldots, Y_n)$$

are in perfect discordance.

H. J. Godwin.

2309. *On Note* 2117.

An integral solution of the equation

$$r^2 + r(x+y) = xy$$

is given by $x=3$, $y=2$, $r=1$. A. F. MACKENZIE.

2310. *The determination of an analytic function of z from a knowledge of its real part without differentiation or integration.*

Suppose that $f(z)$ is an analytic function of the complex variable $z=x+iy$, so that

$$f(z) = u(x, y) + iv(x, y), \quad \text{.................................(1)}$$

where the real functions u and v satisfy the Cauchy-Riemann equations and Laplace's equation. If $\bar{z} = x - iy$, we have the following equalities :

$$g(z, \bar{z}) = u\{(z-\bar{z})/2, (z-\bar{z})/2i\} = \phi(z) + \psi(\bar{z}), \quad \text{..............................(2)}$$

$$ih(z, \bar{z}) = iv\{(z+\bar{z})/2, (z-\bar{z})/2i\} = \phi(z) - \psi(\bar{z}) + C, \quad \text{........................(3)}$$

where C is an appropriate constant. Equations (2) and (3) (identities in z and $\bar{z}$) follow very easily by differentiation. It follows at once that

$$f(z) = 2\phi(z) + C, \quad \overline{f(z)} = 2\psi(\bar{z}) - C. \quad \text{..........................(4)}$$

Example

$$u(x, y) = \log(x^2 + y^2).$$

$$g(z, \bar{z}) = \log(z\bar{z}) = \log z + \log \bar{z},$$

$$f(z) = 2 \log z + iC \quad (C \text{ real}).$$

In many cases, the calculation in the identities (2) and (3) is simplified by giving $\bar{z}$ a special value. Thus, if $\psi(0)$ is finite, we have simply

$$f(z) = 2u(\tfrac{1}{2}z, -\tfrac{1}{2}iz) + A,$$

for an appropriate constant A.

Example

$$u(x, y) = \cos x \sinh y.$$

$$2u(\tfrac{1}{2}z, -\tfrac{1}{2}iz) = -2i \cos \tfrac{1}{2}z \sin \tfrac{1}{2}z = -i \sin z.$$

$$f(z) = -i \sin z + iC \quad (C \text{ real}).$$

If $\psi(z)$ differs from $\phi(z)$ by a constant, we may put $\bar{z} = z$, so that in such a case

$$f(z) = u(z, 0) + C$$

for appropriate C.

Example

$$u(x, y) = e^x \cos y \; ; \quad u(z, 0) = e^z.$$

$$f(z) = e^z + iC \quad (C \text{ real}).$$

In some cases, one can put $\bar{z} = -z$ and then

$$f(z) = u(0, -iz) + C.$$

Example

$$u(x, y) = \exp(x^2 - y^2) \,.\, (x \sin 2xy + y \cos 2xy).$$

$$u(0, -iz) = \exp(z^2) \,.\, (-iz),$$

$$f(z) = -iz \exp(z^2) + C.$$

For polar coordinates, the appropriate substitutions are

$$r=\surd(z\bar{z}),\qquad 2i\theta=\log(z/\bar{z}),$$

so that if $U(r, \theta)$ is the real part of an analytic function, then

$$U\left\{\surd(z\bar{z}),\ \frac{1}{2i}\log(z/\bar{z})\right\}=\phi(z)+\psi(\bar{z}),$$

etc., as in the case of $u(x, y)$.

Reference should be made to the elegant Note 1243 by Milne-Thomson (*Math. Gazette*, XXI (1937), 228–9). A. OPPENHEIM.

2311. *The complex form of the inverse sine.*

Let $\sin^{-1} w=z$, where $z=x+iy$, then

$$w=\sin z,$$

and

$$\surd(1-w^2)=\cos z.$$

Using the conjugate complex numbers w and $\bar{z}$, we have

$$\sin z \sin \bar{z}=w\bar{w},$$

$$\cos z \cos \bar{z}=\surd(1-w^2)\,.\,\surd(1-\bar{w}^2),$$

leading to

$$\cos 2x-\cosh 2y=-2\,|\,w\,|^2,$$

$$\cos 2x+\cosh 2y=2\,|\,\surd(1-w^2)\,|^2$$

which are easily solved for x and y.

Clearly, $\cos^{-1} w$ gives rise to two similar equations and $\sinh^{-1} w$ and $\cosh^{-1} w$ can be dealt with in the same way. A. BUCKLEY.

2312. *All modern conveniences.*

1. Certain problems in books of mathematical puzzles are concerned with the connection of each of m points A_r ($r=1, 2, \ldots, m$) (*e.g.* houses) to each of n points B_s ($s=1, 2, \ldots, n$) (*e.g.* gasworks, waterworks, power stations), all the points lying on a given simple surface, by lines on the surface in such a way that no two of these lines intersect; the most common of these problems is to show the impossibility of a complete connection in the case $m=n=3$. In this note I find the greatest possible number of such lines which can be drawn in the general case ($m, n \geqslant 2$).

2. Call the $m+n$ points (A_r) and (B_s) *vertices*. Let C be a network of lines joining the vertices in such a way that (i) each line of C joins an A point to a B point; (ii) there is at most one line of C joining any two vertices; and (iii) no two lines of C have a point in common except, possibly, end points.

Let $\delta(C)$ denote the number of lines of C, and let Δ be the maximum of $\delta(C)$ over all networks C. Let C' be a network such that $\delta(C')=\Delta$.

3. From § 2 (i), (iii) it follows that C' is topologically equivalent to the network formed by the edges of a polyhedron P having Δ edges and $m+n$ vertices, so that Euler's formula

$$V-E+F=2 \quad \text{.................................(1)}$$

holds, where

$$V=m+n,\quad E=\Delta. \quad \text{.................................(2)}$$

Also, by § 2 (i), the vertices surrounding each face of P are alternately from (A_r) and (B_s), and so each face of P has an even number of sides. By § 2 (ii), it cannot have just two sides, and if it had six or more it would be possible to join two of its vertices, not previously joined by a line of C', by a line in its interior not crossing any line of C', contrary to the hypothesis that C' is a best

possible connection. It follows that each face of P has just four sides, and, as each side is common to just two faces, it follows that

$$4F = 2E. \qquad \qquad (3)$$

Combining (1), (2) and (3), we have

$$\Delta = 2(m+n-2),$$

and so this is the greatest number of lines which can be drawn satisfying the conditions of § 1.

4. The problem admits of two immediate generalisations; let there be p ($\geqslant 3$) sets of points (for instance, n_1 houses, n_2 shops, n_3 wholesalers, etc.) instead of two, and instead of § 2 (i) impose the rule: either, (α) each line of C joins a point of one set to a point of some other set; or, (β) each line of C joins a point of one set to a point of an adjacent set (for example, if the sets are $A_1, A_2, \ldots, A_p$, each line of C joins a point of A_r to a point of A_{r+1} with $A_{p+1} \equiv A_1$).

For (α) it is easy to show that each face of P is a triangle, and so that

$$\Delta = 3(n_1 + n_2 + \ldots + n_p - 2);$$

for (β), with $p=3$, we have a repetition of (α) with $p=3$; for (β), with $p \geqslant 4$, it may be shown that each face of P is a quadrilateral, and so, that

$$\Delta = 2(n_1 + n_2 + \ldots + n_p - 2).$$

This last case may be demonstrated as follows: as $p \geqslant 4$, no face of P can be a triangle, and so if a P can be found all of whose faces are quadrilateral, the corresponding C is a best possible connection. Also, any quadrilateral face of P must have at least one pair of opposite vertices belonging to the same set. As each set has at least two members we can build up a network of faces according to the scheme

A_1	A_2	A_3	A_4	—	A_{p-2}	A_{p-1}
A_2	A_3	A_4	A_5	—	A_{p-1}	A_p

and then add externally to this $2(p-1)$-agon whose vertices are in order $A_1, A_2, \ldots, A_{p-1}, A_p, A_{p-1}, \ldots, A_3, A_2$, quadrilaterals bringing in one more vertex each time without altering the order of the vertices of the bounding $2(p-1)$-agon. When all the vertices have been added in this manner we can add externally lines joining the A_{p-1}, A_{p-2} previously unjoined leaving a $2(p-2)$-agon having no A_p on its boundary, and repeat until the boundary is quadrilateral.

H. ApSimon.

2313. *Etymology of sine.*

The admirable historical summary in the M.A. *Report on the teaching of trigonometry* notes the Arab use of the word "jaib" (bosom), but then translates the Latin word "sinus" as "bay". It is perfectly true that the word has this meaning by extension, but the more usual sense is the bosom of a garment; in other words, "sinus" is an exact translation of the Arabic "jaib".

C. B. Gordon.

2314. *On Note* 2281.

The method of obtaining the expression for the kinetic energy of a body moving in two dimensions, to which Mr. Quadling refers, is given on p. 210 of my *Dynamics* (University Tutorial Press).

S. L. Green.

REVIEWS.

Lectures in Abstract Algebra. I : Basic Concepts. By NATHAN JACOBSON. Pp. viii, 217, 37s. 6d. 1951. (D. van Nostrand, New York : Macmillan, London).

This important book is the first part of a treatise on Abstract Algebra and is to be followed by volumes on *Linear Algebra* and *The Theory of Fields and Galois Theory*.

In the present volume the basic concepts of algebra are introduced and the general theories of groups, rings, fields and lattices are developed systematically from the foundations. No previous knowledge of the subject is assumed, but at one place the reader is expected to be familiar with the elementary properties of determinants of any order. Although the author does not intend to treat exhaustively any of the topics selected, he carries the investigation beyond the most elementary level because he holds that " even at the present stage a deeper understanding of a few topics is to be preferred to a superficial understanding of many ".

The following brief summary of the contents will give an idea of the ground covered. The Introduction deals with the fundamental notions of sets and mappings. In Chapter I semi-groups and groups are defined and the theory is developed as far as the fundamental theorem on homomorphisms. The next three chapters deal with general ring and field theory, including the construction of the field of fractions of a commutative integral domain and a discussion of simple transcendental or algebraic extensions of a field. This is followed by a detailed account of factorization in a commutative integral domain. The properties of a domain being Gaussian (existence of unique factorization) or being Euclidean (existence of a Euclidean algorithm relative to a valuation function) or being a principal ideal domain are analyzed and their mutual relationship is examined. Chapter V contains the most important results about groups with operators, in particular the homomorphism theorems and the Jordan-Hölder and the Krull-Schmidt theorems. The next chapter begins with an account of modules, a discussion of the chain conditions and a proof of the Hilbert basis theorem. This is followed by an exposition of Noetherian rings and of integral dependence. The final Chapter VII is devoted to lattice theory, with emphasis on applications to group and ring theory.

It is evident that this book stresses the abstract approach to algebra, although here and there some concrete material is included such as symmetric polynomials, quaternions and integers in quadratic fields. The author himself admits that the beginner may find the account " at times uncomfortably abstract " and he urges him to study the supplementary exercises and examples in order to consolidate his newly acquired knowledge. But the exposition is so lucid that even such a beginner, if he heeds the author's advice, should not find it too hard to become familiar with the abstract concepts introduced. He certainly cannot fail to appreciate the elegance and beauty of algebraic structures and admire the simple yet powerful methods which algebraists have devised during the last three decades. With consummate expository skill Prof. Jacobson has provided a text-book of abstract algebra suitable for comparatively inexperienced students, who have not previously been acquainted with those concrete facts of classical algebra from which the modern abstract theories have sprung. It is not a bold prediction to affirm that these *Lectures* will exert a powerful influence on the teaching of algebra to Honours students at our universities. Being thoroughly modern in outlook this work is an excellent preparation for the study of contemporary research papers.

The lay-out is very pleasing and the printing accurate. (Attention may be drawn to an error in exercise 1, p. 197, where in the last sentence " first "

should be replaced by " second "). After reading this book the first thought that will come to the mind of most people, as it did to the reviewer, is the hope that the appearance of the remaining two volumes will not be long delayed.

WALTER LEDERMANN.

Fourier Transforms. By I. N. SNEDDON. Pp. xii, 542. 85s. 1951. (McGraw-Hill)

If $f(x)$ is defined by a differential equation and certain boundary conditions, it is sometimes simpler to translate the boundary value problem into one for the function

$$I_f(\alpha) = \int_0^\infty f(x)\, K(\alpha, x)\, dx,$$

where $K(\alpha, x)$ is a known function of α and x. $I_f(\alpha)$ is called the integral transform of $f(x)$ and $K(\alpha, x)$ the kernel of the transform. If x is one of the independent variables in the (partial) differential equation, the effect of translating the equation into one for its transform is to exclude the variable x and leave for solution a differential equation in one less variable. The solution of this equation, which will be a function of α and the remaining independent variables, has then to be " inverted " to recover the " lost " variable. The inversion process means, in effect, solving the above integral equation when $I_f(\alpha)$ is known and $f(x)$ is to be found.

For the boundary value problems of mathematical physics the kernels most commonly used are

$$e^{\alpha x}, \quad {}^{\sin}_{\cos}\, \alpha x, \quad J_\nu(\alpha x), \quad x^{\alpha-1},$$

and the corresponding transforms are known respectively as those of Laplace, Fourier, Hankel and Mellin. Inversion theorems for all these transforms can be set up and the first three chapters (about 90 pages) of the book under review give a careful and thorough account of this basic theory. Since Professor Sneddon has written his book for those whose interest is primarily in the application of the theory, these chapters make no attempt to give the foundations in their most general form. Nevertheless, the main theory is given for a class of functions which is wide enough to include those which normally occur in problems of applied mathematics.

Professor Sneddon includes a discussion of " finite " transforms in which the limits of integration in the integral defining the transform are (a, b) instead of $(0, \infty)$. Although the use of such transforms does not solve problems which are incapable of solution by the classical methods of Fourier or Fourier-Bessel series, it does facilitate their solution. Professor Sneddon has taken a prominent role in advocating the use of these transforms and the method does appear to have a distinct advantage over the classical methods which often require some ingenuity in assuming at the outset the correct form of the solution.

The remaining seven chapters (some 400 pages) are devoted to the application of the theory to problems of mathematical physics and engineering. The subjects treated are vibrations, conduction of heat in solids, the slowing down of neutrons, hydrodynamical problems, applications to atomic and nuclear physics and problems in elasticity. Each chapter is well written and the basic theory is applied to worth-while problems, most of which have been taken from recent research papers. No special knowledge of mathematical physics is assumed and each chapter opens with a clear discussion of the physical foundations and the derivations of the basic equations. Useful appendices give some properties of Bessel functions, approximate methods of calculating the integrals appearing in the solutions, and tables of transforms.

There is no doubt that in this comprehensive book, Professor Sneddon has succeeded in giving an excellent account of a subject in the development of which he has played a leading part. Most existing works on the applications of transform theory confine themselves to the Laplace transform which is but a part of the whole field. Here the whole subject is well covered. Application of the theory of integral transforms has, without doubt, been of great use in mathematical physics and much of the material of the present book could be given with advantage to "honours" undergraduates. Certainly it will be of immense use to all students and research workers interested in the boundary value problems of physics and engineering.

The printing, lay-out and general appearance of the book are all that might be expected of a product of the McGraw-Hill Book Company. Only one criticism comes to mind—the price (85/-). Admittedly it is a large work and presumably a reduction in price would have meant a a reduction of content. Possible ways of doing this seem to lie in rejecting (or giving in less detail) some of the many illustrative applications, or in omitting the physical foundations (available elsewhere) given at the beginning of the last seven chapters. Either course would have removed something of value but it might well have brought the book within the means of many more interested readers.

C. J. Tranter.

Mathematics. A first course. By M. F. Rosskopf, H. D. Aten and W. D. Reeve. Pp. 472. $2.60. 1951. (McGraw-Hill)

The authors, in their preface, say that this book is intended to meet the needs of students in a first course of high school mathematics. It would seem that the intention is to cover the requirements of the first half or two-thirds of the school course. In judging the merits of this book, one cannot avoid doing so from the standpoint of English requirements. Since it is well known that these are somewhat in advance of those in America, allowances should be made for criticisms which are inevitable.

Manufacturers usually give much attention to general finish and external appearance. There is a tendency in the U.S.A. to supply plenty of " chromium plate " in the production of popular textbooks. It goes without saying that the binding is good and that the printing is excellent and kind to the eye. The " chromium plate ", however, extends to the actual text and mode of treatment. A superficial attractiveness is given to the book in order to convince the reader that it is worth having. There is a lavish use of photographic pictures of excellent quality. There are, for example, seven pictures of a ruler, one with inches and tenths, another with inches and sixteenths, and so on. There are four photographs illustrating the use of set squares (or triangles, as they are called) and another seven showing the main applications of the slide rule. Most of these, at least, serve a useful purpose. But we also have a number of full-page photographs which have no direct bearing on anything in the text. There is, for example, a picture of an airport; another of the Mount Palomar telescope; another depicts a boy in a laboratory reading a burette; and so on. In the middle of the section dealing with graphs, the authors break off in order to give an account (taking over 6 pages) of how Mary constructed a poster graph. Of course, pupils find all these and other features interesting and exciting, but they only touch the surface of things. Being dazzled by them, the reader will not readily notice the omissions although some of these may be vital.

The policy of engaging the child's interest becomes a primary consideration and, to some extent, governs what matter is presented. For example, there is a section on air navigation and another on the related topic of composition of velocities. The value of surveying is given as a reason for studying trig-

onometry. On the other hand, where it is not easy to find everyday applications to a topic, there is a tendency to put it into the background or even to omit it. The danger of this policy can be seen when we come to a subject such as proportion. Applications of this are not so obvious.

In the section on graphs we have a few elements of statistics and the pupil is introduced to the meanings of median, mode and mean. The authors appear to be well aware of the growing claims to study this subject. Three chapters on commercial arithmetic are headed: Personal Finances (which includes family budgets, insurance and saving), Government Finances (social security, rates, taxes) and Business (banking, cheques, shop finance). In this arrangement compound interest is discussed two chapters before simple interest, but as only a few simple examples of each are given, it may not turn out as confusing as it seems.

Proportion is not introduced until there is a general discussion of functionality, when it is taken as a particular case of functionality. The pupil is then given a few examples to work out. Thus " If 1 gall. of gasoline costs 17·5c., what is the cost of 3 gall.? " is worked as follows:

$$\text{Let } x = \text{the cost of 3 gall.} \quad \text{(Note!)}$$

$$\therefore \frac{1}{3} = \frac{17{\cdot}5}{x}.$$

The answer is obtained by solving this as an equation. With a simple example on inverse proportion, an equation is obtained after 7 lines of working. It will be realised that there is something very wrong in all this. Functionality, as far as a school treatment is concerned, must rest upon and grow out of a thorough appreciation of the proportion concept. This concept should pervade the whole treatment of school mathematics for it is the cement which binds the various parts into a unified whole.

The handling of geometry causes the reviewer the greatest perplexity. There are two introductory chapters, one of which deals with the use of instruments and nothing else. Basic properties such as angle properties of parallel straight lines and angles at a point are not mentioned. Nothing is said about congruent triangles and it is clear that the pupils are not introduced to even the most elementary example of a proof. We find here, more plainly than elsewhere, the essential difference between the American and English treatments of High School mathematics. The American treatment is not an intellectual approach and avoids any reasoning which is not of the simplest. It would be true to say that this book is written for a type of pupil which is very different from the average in our Grammar Schools and which is much nearer to our Modern School pupil. Modern School teachers would therefore find much that is of interest in this book. S. I.

Über Kurven und Flächen in Allgemeinen Räumen. By P. Finsler. Pp. 170. Sw. Fr. 14.80. 1951. (Birkhäuser, Basel)

This is a reprint of the thesis with the same title presented by Finsler for his doctorate in Göttingen in 1918. The very extensive use which has been made of it is well indicated by the bibliography, (prepared by H. Schubert) which appears at the end, and which includes works up to 1949 which are related to the subject matter of the Finsler thesis.

At the time of its appearance its importance was not immediately realized because Riemannian Geometry, of which what has become known as Finsler Geometry is a generalization, was itself in rather an early stage of development. The notion of parallelism of Levi Civita was only discovered a year earlier, and Weyl's notion of affine connection was still to come. It was

not until 1925 that papers started to appear on Finsler Geometry itself, with the first attempts directed naturally at the extension of the notion of affine connection which had become so dominant in Differential Geometry.

The generalization of ordinary Riemannian Geometry accomplished by Finsler is that of taking an integral as a fundamental invariant instead of a quadratic differential form whose coefficients are functions of the coordinates only. He systematized into a coherent theory certain geometrical notions which had already appeared in works on the Calculus of Variations by Carathéodory, Bliss and Landsberg. The integrand occurring in the fundamental integral involves the coordinates as well as their first derivatives with respect to a parameter, and satisfies certain restrictive conditions which permit its expression as a quadratic form with coefficients depending both on the coordinates and their first derivatives. These derivatives constitute the components of a vector, and quantities occuring in this geometry (tensors, coefficients of connection, etc.) all depend on this vector which is called the " element of support " at each point. Finsler generalized a large number of the usual geometrical notions in his thesis.

Many of the developments from this thesis have arisen from a consideration of a more general " element of support " than a vector. One generalization arises by considering a multiple integral instead of a simple integral, the integrand in that case satisfying such conditions that the integral can represent the area of a portion of surface. In this case area is the fundamental metrical concept, and length a derived concept.

The number of papers still appearing on developments from Finsler's thesis is a justification for this reprint, and its appearance will be welcomed by many workers in this field.

E. T. Davies.

Colloque de Géométrie Différentielle, à Louvain 1951. Centre Belge de Recherches Mathématiques. Pp. 240. 350 fr. Belg; 2,450 fr. French. 1951. (Georges Thone, Liége; Masson & Cie, Paris)

This book contains a series of papers on differential geometry read at the Colloquium of Differential Geometry held at Louvain in April 1951. The President of the Colloquium, Professor Godeaux, explains in an introductory note that as a matter of policy, it was agreed that papers should not be restricted to one topic but that they should range over a very wide field so that many different viewpoints of the subject would be given. That this has been achieved can be seen from the following list of titles :—

Bompiani, E., " Topologie des éléments différentiels et quelques applications ".
Favard, J., " Sur quelques problèmes des couvercles ".
Terracini, A., " La notion d'incidence de plans ' infiniment voisins ' ".
Schouten, J. A., " Sur les tenseurs de V_n aux directions principales V_{n-1} – normales ".
Vincensini, P., " Sur les réseaux et les congruences (ω) ".
Haantjes, J., " Sur la géométrie infinitésimale des espaces métriques ".
Lichnerowicz, A., " Généralisations de la géométrie kählérienne globale ".
Bompiani, E., " Géométries riemanniennes d'espèce supérieure ".
Hlavaty, V., " Géométrie différentielle de contact ".
Kuiper, N. H., " Sur les propriétés conformes des espaces d'Einstein ".
Simonart, F., " Le théorème fondamental de la géométrie textile ".
Van Bouchout, V., " Les lignes hexagonales dans les réseaux de surfaces ".
Backes, F., " La méthode du pentasphère oblique mobile et ses applications ".
Godeaux, L., " Sur les surfaces associées á une suite de Laplace terminée ".

Rozet. O., " Sur les congruences non W de droites."
Debever, R., " Les espaces de l'électromagnétisme ".

Owing to the range of topics covered, it is doubtful whether any single investigator will read more than a selection of these papers. There can be no doubt, however, that this book should be available for reference wherever modern differential geometry is studied. The two papers by Bompiani make very interesting reading ; and the paper by Lichnerowicz is one of the most lucid accounts yet written of the homology theory of certain types of Riemannian manifolds.

The book is very well produced, the printing is good and misprints are not numerous. T. J. WILLMORE.

Einführung in die Funktionentheorie. By L. BIEBERBACH. 2nd edition. Pp. 220. DM. 12.60. 1952. (Verlag für Wissenschaft, Bielefeld)

It is a pleasure to welcome a second edition of Bieberbach's excellent introduction to function theory. This book is clear, lucid, compact and, within its self-imposed limits, comprehensive. It covers approximately the requirements of an undergraduate honours course, and assumes no more than that the reader is familiar with the elementary calculus.

The book is divided into 29 sections or paragraphs. Of these the first five give the definition of complex numbers, simple transformations and an account of the differentiation of complex functions. Complex numbers are defined by the symbol $a+ib$, $i=\sqrt{(-1)}$, (not by ordered pairs (a, b)) and are immediately related to vectors in the Argand (Gauss) diagram. Although there are logical objections to this procedure, it enables the student to become easily familiar with the algebraic manipulations of complex numbers.

The next two sections describe the transformations $w=z^2$, $w=\frac{1}{2}(z-z^{-1})$ and their associated Riemann surfaces. Sections 8 and 9 on series and integration are preparatory to the proof of Cauchy's theorem and Cauchy's integral formula. Section 12 is on series of analytic functions, and the next section, on bounded functions, includes the maximum modulus principle and Schwarz's Lemma.

After paragraphs on the exponential, logarithmic and trigonometric functions the author uses the techniques available to obtain expansions as power series, rational fractions and products. There are interesting sections on the logarithmic residue and converse functions, including Rouché's theorem, and on Vitali's theorem.

Finally there are four sections on conformal transformations followed by one on potential theory and one on hydrodynamical applications.

Almost all the sections include a number of instructive examples. The theorems are stated with an unusual degree of generality but this does not cause any loss of clarity in the exposition. The author has succeeded in compressing a considerable amount of material into a short space, and this conciseness may make some parts of the book difficult for a student to grasp at the first reading.

Although the presentation is not particularly novel, no-one can read this book without being struck by the remarkable beauty of the theory of the complex variable, and for this reason alone it should be warmly recommended to all those who can read German. H. G. EGGLESTON.

Die Fakultät und Verwandte Funktionen. By F. LÖSCH and F. SCHOBLIK. Pp. vi, 205. \$4.03. 1951. (Teubner, Leipzig)

In the foreword to this book it is stated that the intention of the authors is to provide an introduction to the theory of the factorial (or Gamma) function suitable for pure mathematicians, physicists and engineers. They have

admirably succeeded in this aim and done much more besides. In a comparatively small space the authors have contrived to compress practically all the important properties of the factorial function, the incomplete factorial function and their relations with other special functions. A particular feature is the inclusion of a section on applications to various branches of applied mathematics and physics.

The first part of the book deals with the factorial function defined as that solution of the equations

$$f(z+1)=(z+1)f(z), \quad f(1)=1$$

whose reciprocal is an integral function. This definition leads to the Weierstrass product expression for $f(z)$ and thence to the fundamental properties, the asymptotic behaviour and the various expressions of $f(z)$ in terms of series and integrals. The notation used for the factorial function is $z!$ and this is connected with the Gamma function $\Gamma(z)$ by $(z-1)!=\Gamma(z)$. There are also sections giving the evaluation of many definite integrals in terms of the factorial function.

The second part gives a similar development of the properties of the incomplete factorial function $(z, \rho)!$ and the Q-function where

$$(z, \rho)!=\int_0^\rho e^{-t}t^z dt, \quad Q(z, \rho)=\int_\rho^\infty e^{-t}t^z dt$$

and particularly demonstrates the properties of the allied functions the error-function, the integral-exponential function, Hermitian polynomials and the Whittaker function.

In both these parts appropriate emphasis is laid upon the functional equations which the various functions satisfy and this provides a welcome unifying effect. Reference is made to recent work (prior to 1939) and some comparatively new methods and results are included. The exposition is lucid, rigorous, concise and comprehensive.

The third part is devoted to applications in widely diverse fields such as life annuities, radiation from a vertical aerial, thermal conduction, the refractive error in astronomical observations, the banking of a curved railway line on an incline, etc.

This book should be of real value to mathematicians and to those physicists and engineers with a good mathematical background but it is not suitable for those with only a modest mathematical equipment. H. G. EGGLESTON.

Höhere Algebra. I (Lineare Gleichungen); II (Gleichungen höheren Grades). By H. HASSE. 3rd. edition. Pp. 152 and 158. DM. 2.40 each. 1951. Sammlung Göschen Nos. 931 and 932 (de Gruyter, Berlin)

Aufgabensammlung zur Höheren Algebra. By H. HASSE and W. KLOBE. 2nd edition. Pp. 181. DM. 2.40. 1951. Sammlung Göschen No. 1082.

The two volumes of *Höhere Algebra* (first published in 1927) give stimulating introductions to linear algebra and Galois theory. They are notable for their clear and detailed exposition, and for their judicious use of abstract concepts, such as congruence relations and groups, in unifying and illuminating the different aspects of the theories. Some of the short sections (for example, those on groups, the unique factorisation theorem, and finite fields) are particularly good.

Volume I begins with two chapters on fields, groups, etc. The treatment of linear algebra is divided into two parts. In the first, the theorems on the existence of solutions of linear equations are proved by a purely " existence " method due to Toeplitz. In the second, determinants are introduced and then applied to the " practical " solution of linear equations'

The first three chapters of volume II deal with polynomials, algebraic extensions, etc., in preparation for the proof of the main theorem of the Galois theory in chapter IV. The decisive lemmas in this presentation are (a) that a separable extension of finite degree is a stem field (Abel's theorem), and (b) that the degree of a normal extension is equal to the order of its Galois group. Chapter V deals with the solution of equations by radicals.

The present edition differs from the first in that the Galois theory, previously developed only for extensions of a perfect field, is now extended to separable extensions of an arbitrary field. A summary of finite fields has also been added.

The exercises in the Aufgabensammlung are carefully chosen and interesting, and are generally accompanied by hints for solution. In some places the general theory of the " Höhere Algebra " volumes is considerably extended ; for example, the theory of elementary divisors and the basis theorem for finite abelian groups are presented as series of exercises. G. E. WALL.

The theory of functions of a real variable. By R. L. JEFFERY. Pp. xiii, 232. 45s. 1952. Mathematical Expositions, No. 6. (University of Toronto Press; London, Geoffrey Cumberlege)

This new book in the series " Mathematical Expositions " published under the auspices of the University of Toronto has a twofold merit. First, it serves the serious student of Analysis as a thorough and reliable introduction to the theory of functions of one real variable. It is not a textbook of the Calculus and its techniques, but a guide to its fundamental concepts and to a rigorous build-up of its structure. Starting with the axiomatics of the real numbers the book discusses in the earlier chapters I-V the structure of linear point sets, the theory of measure and then that of the Lebesgue integral. The last chapter VIII adds the theory of the Riemann-Stieltjes integral.

Secondly, the intermediate chapters VI and VII treat the delicate question of how far differentiation and integration can be considered as inverse operations. In this field of research Prof. Jeffery has been himself a successful contributor, and the reader will find in these chapters much interesting material (for instance, on non-differentiable continuous functions) that is not easily available elsewhere. Thus this part of the book is an excellent preparation for anyone who intends to work in the field of the modern theory of integration as connected with the names of Denjoy and L. Schwarz.

The book is vividly and well written. The geometrical treatment of the axiomatics of the real numbers (which appear as mere " symbols " attached to points) is perhaps not a fortunate choice for an introduction to Analysis, where Geometry should play rather the role of an intuitive guide than that of an underlying substratum. On the other hand, the reviewer does not share the professed experience of the author that the bare analytical approach through the Lebesgue sums provides for the beginner the easiest access to the understanding of the Lebesgue integral. Here the guiding geometrical connection between area and integral (given by Lebesgue himself as a second approach) seems to be more helpful. However, these are matters of taste and experience, and Prof. Jeffery's treatment is certainly lucid and reliable. An interesting feature of his treatment of measure is that the notion of inner measure is completely avoided. There are many other pleasing tit-bits in his account of the Lebesgue theory : for instance, a proof of the Ergodic Theorem is given as an application. The reviewer also noticed a short and elementary proof of the fundamental theorem on the inversion of the derivative (Theorem 6.6) which seems to be new. The usual textbooks give a rather delicate proof due to Schlesinger and Plessner.

The book can be well recommended as a reliable text for final Honours classes. W. W. ROGOSINSKI.

The Lebesgue Integral. By J. C. Burkill. Pp. viii, 87. 12s. 6d. 1951. Cambridge Tracts, 40. (Cambridge University Press)

There exist, in the English language, full accounts of, and short introductions to, the theory of the Lebesgue integral. Dr. Burkill's new Cambridge Tract does not intend to compete with the former, but it easily outclasses all the latter. True, it does not offer much new material to the expert, but it teaches him how he should teach this subject to others. In fact, the tract is so lucidly planned and so skilfully written that its reading, far from being difficult, will be a delight to anyone with mathematical taste. Again, this introduction is self-contained and yet surprisingly comprehensive in all essentials of the theory. The reader will find, apart from the obvious results, such finer items as Vitali's covering theorem, the theory of absolutely continuous functions, Fubini's theorem, the Riesz-Fischer theorem on strong convergence in L^p, and a concise outline of the theory of the Lebesgue-Stieltjes integral. It is to the reviewer, who has been labouring at a similar task for some time, little short of a miracle how Dr. Burkill succeeded in covering so much, and in so masterly a fashion, on less than 90 pages.

Wisely the author has chosen the geometrical approach to his subject. There are more direct and shorter ways of approach, and more powerful ones, as regards possible extensions of the theory, but they are all severely abstract and less suitable for a first introduction. The primitive geometrical ideas of length, area, and volume come first ; this is also historically the beginning of all integration. Lebesgue's theory of measure is then the satisfactory way of rationalising these ideas. Finally, the integral of a non-negative function is defined, again in a natural way, as the area, or volume, between the range of the function and its graph. Thus, in close contact with familiar geometrical ideas, the whole account of the theory of integration remains lucid and intuitive throughout. The reader will soon appreciate this.

We should be thankful to Dr. Burkill for this tract. It is sad to realise that even now, exactly half a century after Lebesgue's original publications, his theory of integration should still be regarded, in the main, as of interest for the specialist in Analysis only, and not, as it surely is, as a basic domain of knowledge for any serious mathematician, whether pure or applied. Perhaps the existing introductions to the subject were found to be too abstract and forbidding. With this new enjoyable tract available, no one will have such excuse for remaining ignorant. The intelligent reader will find that the new theory is not much more difficult to understand than the classical theory of the Riemann integral. Moreover, he will soon realise that it pays to master it. For, only if we use the Lebesgue integral, the legitimate handling of limiting operations in integral calculus, which caused so much trouble to the older analysts, becomes both easy and intelligible. This is the main reason why it has rightly superseded the older definitions. It is high time that the knowledge of the Lebesgue integral became a more integral part of mathematical education at our Honours Schools.

W. W. Rogosinski.

An introduction to the theory of mechanics. By K. E. Bullen. 2nd edition. Pp. xvi, 368. 21s. 1952. (Cambridge University Press)

Teachers of mechanics will be grateful to the Cambridge Press for publishing in this country a corrected edition of Professor Bullen's book ; the first edition was published in Sydney, and a full and appreciative notice appeared in the *Gazette*, XXXIV, pp. 151-3 (May, 1950).

The contents cover roughly the requirements of the London General B.Sc. (External) for two-dimensional statics and dynamics, and hydrostatics. The most noteworthy features of the book are its freshness and vitality, and the amount of comparatively informal comment, which should be of great assist-

ance to the student reading on his own and to the young teacher. The treatment is generally but not exclusively vectorial, the worked examples are explained with great care, the supply of exercises for the reader is adequate in quantity and variety. T. A. A. B.

Mathematics for technical students. I. II. III. By J. D. N. GASSON. Pp. xii, 417 ; x, 431 ; xii, 451. 15s. ; 15s. ; 18s. 1951. (Cambridge University Press)

The author states that the aim of the books is to provide a complete course of study in Mathematics for the Ordinary National Certificates in Engineering and he must be congratulated at once on refraining from calling it " Practical " Mathematics. It is regretted that his copious examples do not demonstrate as forcefully the author's desire to shake off the shackles of the examinations in so called " Engineering " Mathematics held during the first twenty years of the century. Judging by recent books at this stage, consisting only of problems, the principal examining bodies give ready access to their examination papers to authors, and these books contain problems possessing a variety and refreshing originality that is lacking in Mr. Gasson's work.

However, Mr. Gasson has made a most worthy and conscientious attack on a task terrifying in its formidability. While many have failed within the last twenty years, Mr. Gasson's work should find a place in every Technical College Library, on every Technical College Lecturer's bookshelf and in the brief cases of most of the conscientious students, particularly those who, having left College, have become Technical Assistants.

The explanatory portions are so full as to render the lecturer almost redundant and, in some cases, to embarrass him. This would seem unnecessary, as by the rules of the Ordinary National Certificate course, private students cannot qualify to sit for the examinations.

The embarrassment will occur in several places where Mr Gasson has underestimated the intelligence of his public. These students may not have scintillated academically at school, but having gone to work, the poorest develop a zest and alertness. For instance, too frequently the author, in formula manipulation, tells them the answer instead of asking for a formula for x (say).

Again, when giving worked examples, it is well to be sure that the best method (within the scope of the students) is given. In Book III, page 257, Ex. 7, most students would start with

$$V=\tfrac{1}{3}\pi r^2 h=\tfrac{1}{3}\pi h(l^2-h^2) \quad \text{and get} \quad l=h\sqrt{3},\ \cos\theta=1/\sqrt{3}.$$

On page 247, few would fail to appreciate with glee the joy of substituting for r^2 rather than h. In any case they should appreciate that the maximum of $\tfrac{1}{2}r^2\sqrt{(9-r^2)}$ occurs at the same place as that of $9r^4-r^6$.

The virtue to these students of work like Example 3 on page 244, Book III, is very doubtful. Mr. Gasson rightly recognises the importance of graphs and his work on them is admirable. The adherence to the old fashioned introduction to trigonometry is surprising, involving simultaneously the loss of the introduction to vectors and projection and an increased tax on memory. Also included is the terrible mnemonic " all, sine, tan, cos " for the signs of the ratios in the various quadrants. Full as are the explanations, the introductions to the two branches of calculus are very disappointing. Far too many of our students are too eager to tell our critics that " If $y=x^2$, $\dfrac{dy}{dx}=2x$ but we haven't a clue what it is about ". Hence the vital importance to all teachers of the introductions.

The task undertaken by Mr. Gasson is intensified in difficulty by the varied syllabuses in our Colleges ; Mr. Gasson overcomes this by including everything. He has therefore produced a useful encyclopaedia but can be charged with being too academic. His algebra problems about speeds of trains will have a

familiar unpleasant ring of school about them : his students, now " grown up ", prefer boilers, screwthreads and templates.

The books are beautifully produced and the effect is so striking that it is a pleasure to read them and to use them. There are occasional unfortunate printings $\left(\text{III, p. 352, } \frac{2r \sin \alpha/2}{\alpha}\right)$ while in the figure 42, page 75, Book II, the shading of the plane ACD is disconcerting.

My chief complaint is the price : will the apprentice of 1952 spend £2 8s. 0d. on his Mathematics text book in the first three years? A. J. L. AVERY.

Mathematics at the fireside. By G. L. S. SHACKLE. Pp. xii, 156. 16s. 1952. (Cambridge University Press)

This book is an interesting attempt to explain to " quite young people " a number of the basic ideas on which mathematics is founded. These ideas range from one-to-one correspondence at the start, through limits and continuity, integration and complex numbers to permutations and mathematical induction at the end.

The book takes the form a of a series of dialogues between two children, George and Lucy, and George's Father. This has the advantage that repetition " to a degree that could not otherwise be borne " becomes possible, the children—one or both—repeating what the father has said, or the father repeating and amplifying the children's conclusions. It may be said at once that these repetitions do not seem excessive and that the language throughout is commendably simple. Also that the range of topics covered is remarkable and that all the knowledge reached is of high mathematical value.

It is in the difficult task of producing consistency and verisimilitude that these dialogues are most open to criticism. In the first dialogue the children might be eight to ten, and in the picture they certainly do not appear older than ten to twelve, but later on they need either to be more mature or of remarkable intelligence, and show an exceptional capacity of expressing their meaning clearly. When an idea has been discussed and the suitable technical term introduced, the children seem to have no difficulty in afterwards using that technical term in a manner entirely correct. Schoolmasters will wish that their own pupils could show equal capacity.

Probabilities are not always observed. The reader will often feel that a blackboard is indicated, though the writing is said to be on paper, and he may feel surprised at squared paper being taken on picnics.

One or two small blemishes could easily be corrected in a second edition. On p. 46 there seems no object in writing down $y=\frac{1}{2}(e^x+e^{-x})$, which must have seemed completely incomprehensible to George and Lucy, and then saying that it is called $y=\cosh x$ for short. The second statement, leading to a set of values taken from a table, would have done better by itself. On p. 73 the same letter g is used both in the sense connected with gravity and as a very small change in t.

However these are minor matters. It is to be hoped that the reader is prepared to forget that the children are showing a most improbable capacity for lucid expression. If he can do this he will find many important mathematical ideas clearly explained in simple language.

The book should find a place on the school-room shelves and it is to be hoped that boys and girls will feel tempted to borrow it from the school library.

C. O. TUCKEY.

Algèbre des ensembles. By WACLAW SIERPINSKI. Pp. 205. $4.50. 1951. Monografje Matematyczne, 23 (Warsaw ; Stechert-Hafner, New York)

A book on the theory of sets by a leading Polish mathematician is a promise

of pleasure which few students of mathematics will resist, and Professor Sierpinski's book will disappoint no one. It is beautifully written in a warm teaching style, a mine of information, as rich in classical theorems as in references to recent work.

The first chapter contains a detailed development of the calculus of propositions and the predicate calculus. The propositional calculus is first set up by the valuation or matrix method in which there are no axioms, each true formula taking the value unity for every substitution of zero or unity in its propositional variables. For instance, if implication, denoted by $\to$, is defined by the valuations

$$(1\to1)\equiv1,\ (0\to1)\equiv1,\ (0\to0)\equiv1 \text{ and } (1\to0)\equiv0$$

then the formula

$$p\to\{(p\to q)\to q\}$$

is proved by the equivalences

$$\begin{aligned}1\to\{(1\to1)\to1\}&\equiv1\to(1\to1)\equiv1\to1\equiv1,\\0\to\{(0\to1)\to1\}&\equiv0\to(1\to1)\equiv1\to1\equiv1,\\0\to\{(0\to0)\to0\}&\equiv0\to(1\to0)\equiv0\to0\equiv1,\\1\to\{(1\to0)\to0\}&\equiv1\to(0\to0)\equiv1\to1\equiv1.\end{aligned}$$

Sierpinski denotes disjunction and conjunction by sum and product signs respectively, by contrast with the German logicians, to draw out the analogy with the algebra of classes; in the third chapter the striking similarity between the propositional calculus and the theory of sets is shown to be no accident but an intrinsic correspondence since both the propositional calculus and the theory of sets are interpretations of a Boolean algebra. The use of unity and zero as signs of a true and a false proposition respectively (instead of the converse as in recursive number theory) is of course bound up with the special use of multiplication and addition since $p+q\equiv0$ only in the case $p\equiv0$, $q\equiv0$; and $p\,.\,q\equiv0$ only if one (at least) of p, q is equivalent to zero. It is perhaps worth noting in this connection that Sierpinski's use of 0, 1 is ambiguous since 0, 1 are used both as specimen false and true propositions and also as the *logical values* " false " and " true "; it would seem preferable to omit the reference (p. 2) to logical values.

As an alternative to the valuation basis Lukasiewicz's axiomatic foundation of the propositional calculus is described; in this formulation negation is denoted by Np, implication by Cpq and there are three axioms:

1. $CCpqCCqrCpr$, 2. $CCNppp$, and 3. $CpCNpq$,
(or in German notation

$$(p\to q)\to\{(q\to r)\to(p\to r)\},\quad (\sim p\to p)\to p,\quad p\to(\sim p\to q)\ )$$

Chapter I concludes with an account of the first order predicate calculus and the universal and existential quantifiers. A number of relationships between the quantifiers are given (on p. 31) but there is no indication which of these are postulated and which are derivable.

The second and third chapters are devoted to the classical theory of sets and include a full account of the famous antinomies. Sierpinski does not adopt the theory of types but seeks to protect his set theory from contradiction by requiring that a class be constructed only from elements which may be defined without reference to the class itself. Thus, for instance, the universal class is excluded since one of its members would be the class itself. On the definition of the class concept Sierpinski's attitude is perhaps best described as fatalistic; he regards a class as defined by a membership condition but recognises that there may be objects of which we are unable to decide whether they are members or not. This possibility is illustrated by a variety

of examples ; it is not known, for instance, Sierpinski observes, whether or not Euler's constant γ belongs to the class of rationals, or whether unity is a member of the class which consists of the two numbers 1, 2 if Fermat's last theorem is true and of the numbers 2,3 otherwise. Likewise we do not know whether the class of all natural numbers n for which 2^{n+16} is prime is empty or not. In all these examples the lack of information has a temporary air, and one may well consider that one day the expansion of knowledge will rob them of their point, but it is easy to construct an example which is free from this objection. Thus if $f(x)$ is the primitive recursive function which Gödel discovered and which is such that a proof (in recursive number theory) either of the universal statement $\underset{x}{\Pi}(f(x)=0)$ or of its contrary $\underset{x}{\Sigma}(f(x)\neq 0)$, leads to a contradiction then the class $\underset{x}{E}(f(x)\neq 0)$ can neither be proved empty nor proved not empty.

Both in these and later chapters many remarkable equivalents of the axion of choice are stated ; one of the simplest of these is Zorn's lemma, which says that in every closed family of sets there is at least one set which is not contained in any other set of the family, and another is that of any two non-empty sets one may be mapped on the other by a single-valued function. Though Gödel's proof that the theory of sets, including the axiom of choice, is free from contradiction if only the system formed by the remaining axioms is free from contradiction, robs the axiom of choice of much of its former importance, it in no way diminishes the interest and value of finding equivalents of the axiom.

The fourth chapter, on single-valued and many-valued mappings, contains a host of exciting recent results. There are Mlle. Piccard's theorems that amongst the $n!$ one-one transformations of a finite set into itself there are two from which all the remaining transformations may be derived by a finite number of repetitions, and that amongst the n^n single-valued functions (of one variable) whose arguments and values belong to a given set of n members, there are 3 functions (and not less than 3 if $n>1$) from which all the rest may be derived by substitution. Sierpinski gives also a new proof of a theorem due to Donald Webb that, given any $m>1$, there is a function of two variables, whose arguments and values are positive integers not exceeding m, from which every function of any number of variables (with values and arguments integral and not exceeding m) may be obtained by substitution. Towards the end of this chapter there is an important section in which topology is formulated as a branch of general set theory, and generalisations of two familiar results in the classical theory of functions are proved ; the first of these is that a decreasing chain of non-empty closed compact sets has a non-empty product and the second is that if a closed compact set is contained in an infinite sum of open sets then it is contained in the sum of a finite number of these sets. The chapter concludes with an application of Cantor's diagonal process to prove that if a function f maps each element t of a set T on a subset $f(t)$ of T, and if E is the set of elements t which are not members of the subset $f(t)$, then E is not a value of the function f.

The fifth (and last) chapter is a study of families of sets. Denoting by Φ' the family of sets which either belong to a family Φ or are the difference of two members of Φ, a characteristic theorem of this chapter is that if both the sum and the product of any two sets of Φ belong to Φ then Φ'' is the family of all the sums of a pair of sets from the family Φ'.

The book is well provided with examples ; some of these (particularly in the first half) are straightforward exercises, but many others are extensions of the text and are either followed by their solution or carry a reference to an original article.

R. L. GOODSTEIN.

Tensor Analysis : Theory and Applications. By I. S. Sokolnikoff. Pp. ix, 335. 48s. 1951. (New York: John Wiley and Sons Inc.; London: Chapman and Hall)

Recently there have appeared in steady succession a number of books on the tensor calculus technique, and the present book is a worthy but expensive member of this sequence.

The author, who is well known in America as a distinguished applied mathematician, has maintained a high standard of rigour in his lectures on which this book is based. The book opens with a lucid account of the pure technique to which two chapters are devoted. In the first the author begins with the simplest ideas of vector analysis and develops the theory of linear vector spaces. Particular care is devoted to defining spaces of different dimensions. The theory of linear independence of a set of vectors leads naturally to the study of linear transformations and matrices, and the first chapter includes a useful account of matrix theory and quadratic forms and the reduction of the latter. In the second chapter the theory of tensor analysis proper is developed, and emphasis is placed on the broader characterisation of tensors by the isomorphism of transformations of coordinates and transformations induced on sets of functions, due to Weyl and Veblen. This chapter is liberally interspersed with useful examples to be worked by the student.

The third chapter is devoted to the application of tensor methods to geometry. The ground covered is that customarily associated with classical differential geometry, except that the tensor treatment is adopted throughout. The choice of topics is partly governed by the need to develop those portions of the subject which are relevant to dynamical theory and the theory of the deformation of plates and shells.

The remaining three chapters of the book concern applied mathematics. The essential concepts of analytical dynamics based on Lagrange's equations are developed by tensor methods in the fourth chapter. The principle of least action is carefully discussed, and several standard theorems of Newtonian gravitation are established. In the fifth chapter the subject of relativistic mechanics is developed, first for the Special Theory of Relativity and then for the General Theory. The author comments on the latter as follows: "... its mathematical elegance and success in explaining the advance of the perihelion of Mercury gave hope that the time when all mathematical physics would be imbedded in the framework of the general theory of relativity was not too far away. However, the researches of the following two decades make it appear unlikely that general relativity will prove useful in the domain of microscopic physics, because of the failure of the theory to unify mechanics and electrodynamics." The author therefore does not waste time in discussing the many abortive attempts to develop a unified field theory, but contents himself with a brief but commendably clear development of General Relativity in so far as it explains the residual perihelion motion of Mercury.

In the sixth and final chapter the author comes to what he clearly regards as the most fruitful field in applied mathematics for applying tensor methods, namely the mechanics of continuous media (in the most general tensor form). He justly claims that this treatment forms a "substantial introduction to non-linear mechanics of fluids and elastic solids". The treatment is based on the work of F. D. Murnaghan on the analysis of deformation of a continuum of identifiable material particles. In his discussion of stress-strain relations the author abandons the classical approach and makes no *ad hoc* assumptions. Instead, he uses thermodynamical principles to deduce a law which includes the classical stress-strain relationship as a special case. The book concludes with a brief discussion of fluids and the derivation of the Eulerian and Navier-Stokes hydrodynamical equations.

The author provides many useful references to other works and original papers. The book is clearly printed and well produced. G. J. W.

Classical Mechanics. By D. E. Rutherford. Pp. viii, 200. 10s. 6d. 1951. (Oliver and Boyd, Edinburgh and London)

This latest volume in the well-known University Mathematical Text Series, edited by Professor Aitken and the author, provides a logical development of classical mechanics beginning with Newton's laws and culminating in Hamilton's Principle. If the book had appeared in the early days of the series it would presumably have been sold at about half its present price, in which case it would, like the earlier volumes in the series, have been outstanding as a standard text-book which (almost) every student could afford. With the present rise in the costs of production the price charged, although very reasonable, is not much less than that of some of the older standard text-books on mechanics, and so comparison with these must now be made more or less irrespective of financial considerations.

Like the other texts in the series the format is pleasing, the print and diagrams clear and the book is compact and can easily be slipped into the pocket. Vector methods are used when required, but the author does not make a fetish of them. Unfortunately, he uses the $\times$ sign instead of $\wedge$ to denote vector product, but although I think this is a pity it is of minor consequence. A more serious point arises out of the book's main object. The author makes a praiseworthy attempt to present the subject as a logical discipline without losing sight of its important physical applications. This is a vast improvement over many older introductions to the subject, but the present approach is not one that the beginner (even at the university level) would be best advised to follow without expert advice. For example, on page 13 we are already plunged into the sophisticated topic " rate of change of a vector moving relative to moving axes ", whereas not until page 76 do we come to a discussion of the simple pendulum. It is clear that the tyro will need a supervisor to guide him on the order in which he reads the various sections of this book, a point which the author himself explicitly recognises. A more advanced student, however, will benefit when revising the subject by seeing it presented in a truly logical fashion. In one respect the book seems to be inadequate, namely in its treatment of the concept " angular velocity of a rigid body ". The fundamental existence theorem seems to me to be omitted (contrast the treatment in E. A. Milne's *Vectorial Mechanics*).

The book falls into five chapters ; Kinematics, The Nature of Force, Dynamics of a Particle, Dynamics of Rigid Bodies, Generalised Coordinates. The author regards the " special feature of the book " to lie in the chapter on force. This chapter forms a reasonably good brief introduction to the subject, treating it in a unified fashion while recognising that there are points of difficulty in the very foundations which are often slurred over.

The author is to be congratulated on having cut out much of the " dead wood " which has come to be associated with the traditional teaching of dynamics, and his claim that " several examples of physical significance have been included " is over-modest. For instance, he has some good examples on collisions of fundamental physical particles in place of the stock billiard ball questions. Incidentally, any teacher of this topic is well-advised to consult also the beautiful, and essentially elementary, book *Nuclear Collisions in Photographs* by Powell and Occhialini (1947, O.U.P.). With its aid the whole subject of elastic collisions comes alive.

There are in all over one hundred examples to be worked by the reader. They are arranged in sets attached to each of the five chapters. The book is strongly recommended for students with tutors to advise them. G. J. Whitrow.

Les récréations mathématiques. (Parmi les nombres curieux). By V. Thébault. Pp. vi, 297. 2500 fr. 1951. (Gauthier-Villars)

Neither author nor topic needs much introduction to readers of the *Gazette*. Professor Thébault has contributed to our pages many items on geometry and arithmetic, marked by ingenuity and often by a somewhat deceptive simplicity. His topic is so vast that few mathematicians can fail to have touched on it at some point, and its interest is never-failing. When Johnson was in the Hebrides, he presented to the daughter of his host a book which he happened to have with him. On being twitted by Boswell on " happening " to have with him a copy of Cocker's *Arithmetic*, Johnson's annihilating reply was : " Why, Sir, . . ., a book of science is inexhaustible."

Professor Thébault's purpose is not solely to amuse us. He points out that these problems may require for their solution as much insight, ingenuity and sublety of artifice as those which occur in more advanced mathematics, and so provide an admirable intellectual exercise ; he is not afraid to quote Jacobi : " Le but unique de la science, c'est l'honneur de l'esprit humain, et, sous ce titre, une question des nombres vaut autant qu'une question du système des mondes." Curiosities may have curious consequences : Fermat's Last Theorem is in itself little more than a curiosity, but from it came the theory of ideals and from that a weapon of immense power in modern algebraic geometry.

It would be unwise to insist too much on an ulterior motive ; rather let us enjoy the rich collection made from Professor Thébault's contributions to many periodicals. How many perfect squares can be formed from the digits 1, 2, 3, ..., 9, each taken once? Professor Thébault gives us the thirty solutions. In what scales of notation is it impossible to have a perfect square whose digits (two at least) are all odd? The exceptional scales are those whose bases are 2, 4, 10, 12. Those who like to have exercises to practice on may turn to the " 100 problèmes variés ", and find, for instance, those numbers which are three times the square of the sum of their digits : answers and sometimes hints are given, the answer to this problem being 243 and 972. Since many of the problems discussed in the book relate to perfect squares, the table of squares of integers from 1 to 1000 in the scales of notation whose bases are 2, 3, 4, 5, 6, 7, 8, 9, 11, 12 will be found helpful.

Apart from its entertainment value, I can imagine that this volume in the hands of a bright boy might possess much worth as a stimulus : certainly it well deserves a place on the shelves of the school library. The prerequisite equipment is small, the results are clearly stated and lucidly expounded, and the Gauthier-Villars printing, as always, excellent. T. A. A. B.

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